Problem 12260
(American Mathematical Monthly, Vol.128, June-July 2021)

Proposed by S. M. Stewart (Australia).

Prove

$$\int_0^\infty \frac{\sin^2(x) - x\sin(x)}{x^3}\,dx = \frac{1}{2} - \log(2).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. After integrating by parts twice, we have that

$$\begin{aligned}\int \frac{\sin^2(x) - x\sin(x)}{x^3}\,dx &= -\frac{\sin^2(x) - x\sin(x)}{2x^2} + \int \frac{\sin(2x) - \sin(x) - x\cos(x)}{2x^2}\,dx \\ &= -\frac{\sin^2(x) - x\sin(x)}{2x^2} - \frac{\sin(2x) - \sin(x)}{2x} + \int \frac{\cos(2x) - \cos(x)}{x}\,dx \\ &= -\frac{\sin^2(x)}{2x^2} + \frac{\sin(x)}{x} - \frac{\sin(2x)}{2x} + \int \frac{\cos(2x) - \cos(x)}{x}\,dx.\end{aligned}$$

Moreover, for $0 < r < R$,

$$\begin{aligned}\int_r^R \frac{\cos(2x) - \cos(x)}{x}\,dx &= \int_{2r}^{2R} \frac{\cos(x)}{x}\,dx - \int_r^R \frac{\cos(x)}{x}\,dx \\ &= \int_R^{2R} \frac{\cos(x)}{x}\,dx - \int_r^{2r} \frac{\cos(x)}{x}\,dx.\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^\infty \frac{\sin^2(x) - x\sin(x)}{x^3}\,dx &= \left[-\frac{\sin^2(x)}{2x^2} + \frac{\sin(x)}{x} - \frac{\sin(2x)}{2x}\right]_{0^+}^\infty \\ &\qquad + \lim_{R\to\infty}\int_R^{2R} \frac{\cos(x)}{x}\,dx - \lim_{r\to 0^+}\int_r^{2r} \frac{\cos(x)}{x}\,dx \\ &= \frac{1}{2} + \lim_{R\to\infty}\left[\frac{\sin(x)}{x}\right]_R^{2R} + \lim_{R\to\infty}\int_R^{2R} \frac{\sin(x)}{x^2}\,dx - \lim_{r\to 0^+}\int_r^{2r} \frac{\cos(x)}{x}\,dx \\ &= \frac{1}{2} + 0 + 0 - \log(2) = \frac{1}{2} - \log(2)\end{aligned}$$

where we used the bounds:

$$\left|\int_R^{2R} \frac{\sin(x)}{x^2}\,dx\right| \leq \int_R^{2R} \frac{1}{x^2}\,dx = \frac{1}{2R},$$

and, for $0 < 2r < \pi/2$,

$$\log(2) - \frac{3r^2}{4} = \int_r^{2r} \frac{1 - \frac{x^2}{2}}{x}\,dx \leq \int_r^{2r} \frac{\cos(x)}{x}\,dx \leq \int_r^{2r} \frac{1}{x}\,dx = \log(2).$$

□