Problem 12259
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Proposed by G. Fera (Italy).

A triangle is Heronian if it has integer sides and integer area. A pair of noncongruent Heronian triangles is called a supplementary pair if the triangles have the same perimeter and the same area and some interior angle of one is the supplement of some interior angle of the other. Prove that there are infinitely many supplementary pairs of Heronian triangles.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. For any integer $n \geq 2$, let T_1 be the triangle with side lengths

$$(a_1, b_1, c_1) = ((n^2+1)^2, n^2(n^4+n^2+1), n^6+2n^4+n^2+1),$$

and let T_2 be the triangle with side lengths

$$(a_2, b_2, c_2) = (n^2(n^2+1)^2, n^4+n^2+1, n^6+n^4+2n^2+1).$$

It is straightforward to check that T_1 and T_2 are non congruent Heronian triangles which have the same perimeter and the same area:

$$\begin{cases} \text{perimeter}(T_1) = \text{perimeter}(T_2) = 2(n^2+1)(n^4+n^2+1), \\ \text{area}(T_1) = \text{area}(T_2) = n^3(n^2+1)(n^4+n^2+1). \end{cases}$$

Morever, let s_i be the half-perimeter of T_i for $i = 1, 2$, then

$$\tan\left(\frac{C_1}{2}\right) = \sqrt{\frac{(s_1-a_1)(s_1-b_1)}{s_1(s_1-c_1)}} = n \qquad \text{and} \qquad \tan\left(\frac{C_2}{2}\right) = \sqrt{\frac{(s_2-a_2)(s_2-b_2)}{s_2(s_2-c_2)}} = \frac{1}{n}$$

which implies that the angle C_1 is the supplement of C_2.
Hence, by choosing any integer $n \geq 2$, we generate an infinite family of supplementary pairs (T_1, T_2) of noncongruent Heronian triangles. □