Problem 12258
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Let S be the set of positive integers n such that $n!$ is not the sum of three squares. Show that S has bounded gaps, i.e., there is a positive constant C such that for every positive integer n, there is an element of S between n and $n+C$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Any positive integer can be written in a unique way as the product of a power of two 2^a and an odd positive integer d. By Legendre's three square theorem, a positive integer can not be represented as the sum of three squares of integers if and only $a \equiv 0 \pmod 2$ and $d \equiv 7 \pmod 8$.

We claim that for $C = 64 + 14 = 78$, for every positive integer n there is and element of S in the interval $[n, n+78)$.

Indeed, by letting $q = \lceil n/64 \rceil$, we have that $n \leq 64q < 64q + 14 < n + 78$. Moreover

$$(64q+r)! = (64q)! \cdot \prod_{j=1}^{r}(64q+j) = (2^a \cdot d) \cdot (2^{a_{q,r}} \cdot d_{q,r}) = 2^{a+a_{q,r}} \cdot (d \cdot d_{q,r})$$

According to the table below, $a_{q,r} \pmod 2$ and $d_{q,r} \pmod 8$ does not depend on q, and we can always choose a suitable $0 \leq r \leq 14$ such that $a + a_{q,r} \equiv 0 \pmod 2$ and $d \cdot d_{q,r} \equiv 7 \pmod 8$, that is $64q + r \in S$.

r	$a_{q,r} \pmod 2$	$d_{q,r} \pmod 8$
0	0	1
7	0	3
6	0	5
10	0	7
2	1	1
3	1	3
14	1	5
5	1	7

□