Problem 12257
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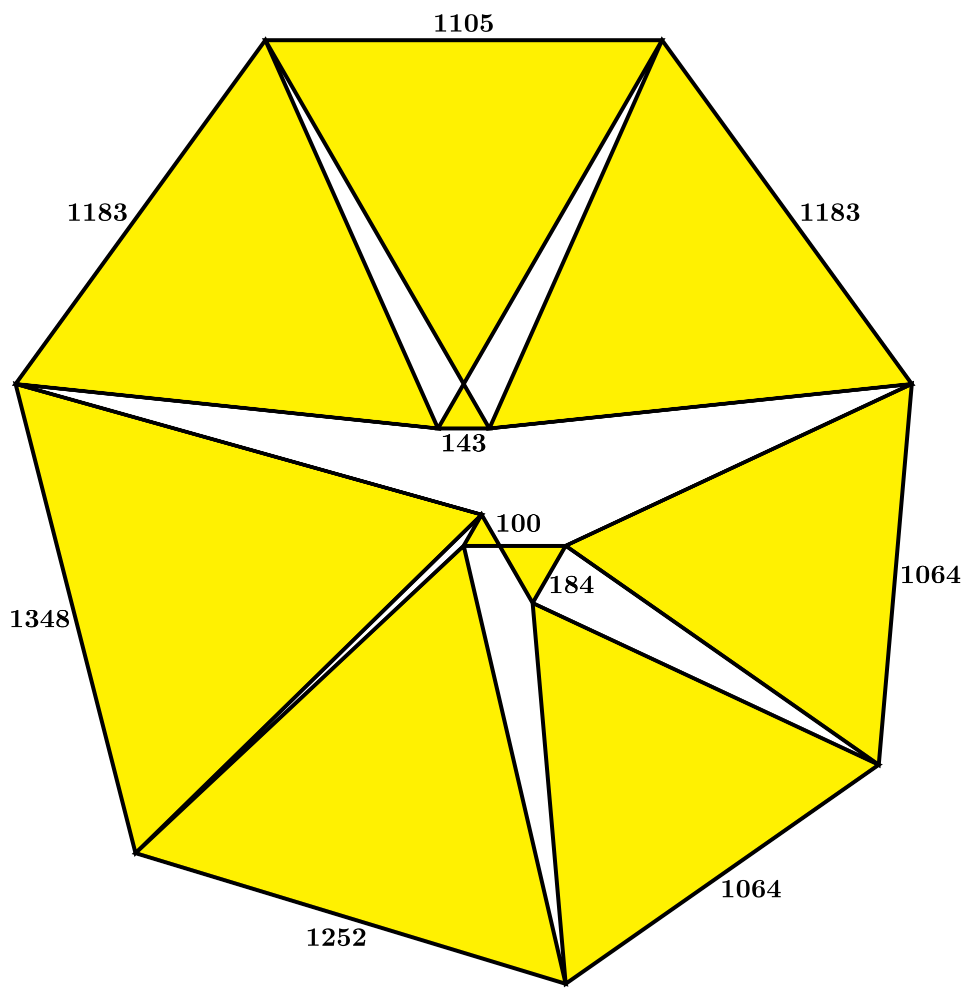
Proposed by E. Friedman (USA) and J. Tilley (USA).

An arrangement of equilateral triangles in the plane is called saturated if the intersection of any two is either empty or is a common vertex and every vertex is shared by exactly two triangles. What is the smallest positive integer n such that there exists a saturated arrangement of n equilateral triangles with integer length sides?

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We claim that the smallest positive integer is 10. Below we give an example of an arrangement of 10 saturated equilateral triangles with integer sides (the lengths are specified beside each triangle). It remains to show that there is no arrangement with less than 10 triangles.

We first note that since every vertex is shared by exactly two triangles, then an arrangement of n saturated triangles has $3n/2$ vertices and therefore n has to be even. So we may assume by contradiction that $n \leq 8$. Moreover, the boundary of the unbounded region of an arrangement (which is 4-regular planar graph) is a polygon with b sides. These sides are the sides of b equilateral triangles pointing inside. The intersection of each pair of adjacent triangles along this boundary is just a vertex which implies that each internal angle of the polygon is more than 120° and therefore $n \geq b > 6$. This means that $n = 8$ and $b \in \{7, 8\}$. As regards the b internal vertices, we may try to join some pair but, in any case, it is easy to verify that in order to cover the remaining vertices we need more than one new triangle. Hence $n > b + 1 \geq 7 + 1 = 8$ and we have a contradiction.



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