Problem 12256
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Proposed by P. Bracken (USA).

Prove

$$\int_0^1 \frac{\log(1+x)\log(1-x)}{x} = -\frac{5}{8}\,\zeta(3).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We first note that

$$\log^2((1+x)(1-x)) - \log^2\left(\frac{1+x}{1-x}\right) = 4\log(1+x)\log(1-x).$$

Then

$$\begin{aligned}\int_0^1 \frac{\log(1+x)\log(1-x)}{x}\,dx &= \frac{1}{4}\left(\int_0^1 \frac{1}{x}\log^2(1-x^2)\,dx - \int_0^1 \frac{1}{x}\log^2\left(\frac{1+x}{1-x}\right)dx\right)\\ &= \frac{1}{4}\left(\zeta(3) - \frac{7\zeta(3)}{2}\right) = -\frac{5}{8}\,\zeta(3)\end{aligned}$$

because, by letting $t = 1 - x^2$, we get

$$\begin{aligned}\int_0^1 \frac{1}{x}\log^2(1-x^2)\,dx &= \frac{1}{2}\int_0^1 \frac{\log^2(t)}{1-t}\,dt = \frac{1}{2}\sum_{k=0}^{\infty}\int_0^1 t^k \log^2(t)\,dt\\ &= \frac{1}{2}\sum_{k=0}^{\infty}\frac{2}{(k+1)^3} = \zeta(3),\end{aligned}$$

and, by letting $t = \frac{1+x}{1-x}$, we obtain

$$\begin{aligned}\int_0^1 \frac{1}{x}\log^2\left(\frac{1+x}{1-x}\right)dx &= 2\int_0^1 \frac{\log^2(t)}{1-t^2}\,dt = 2\sum_{k=0}^{\infty}\int_0^1 t^{2k}\log^2(t)\,dt\\ &= 2\sum_{k=0}^{\infty}\frac{2}{(2k+1)^3} = \frac{7}{2}\,\zeta(3).\end{aligned}$$

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