Problem 12254
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Prove

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1}\sum_{k=1}^{n}\frac{1}{n+k}\right)=\frac{3\pi}{8}\log(2)-G$$

where G is Catalan's constant $\sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $H_n=\sum_{j=1}^{n}1/j$, then

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1}\sum_{k=1}^{n}\frac{1}{n+k}\right)&=\sum_{n=0}^{\infty}\frac{(-1)^nH_{2n}}{2n+1}-\sum_{n=0}^{\infty}\frac{(-1)^nH_n}{2n+1}\\&=\left(-\frac{\pi\log(2)}{8}\right)-\left(-\frac{\pi\log(2)}{2}+G\right)=\frac{3\pi}{8}\log(2)-G\end{aligned}$$

where the two series are evaluated below.
1) For $x\in[0,1]$,

$$\begin{aligned}\arctan(x)\log(1+x^2)&=\sum_{k=0}^{\infty}\frac{(-1)^kx^{2k+1}}{2k+1}\sum_{k=0}^{\infty}\frac{(-1)^{k-1}x^{2k}}{k}=\sum_{n=1}^{\infty}x^{2n+1}(-1)^{n-1}\sum_{k=0}^{n-1}\frac{1}{(2k+1)(n-k)}\\&=-\sum_{n=1}^{\infty}\frac{x^{2n+1}(-1)^n}{2n+1}\left(\sum_{k=0}^{n-1}\frac{1}{n-k}+2\sum_{k=0}^{n-1}\frac{1}{2k+1}\right)=-2\sum_{n=1}^{\infty}\frac{x^{2n+1}(-1)^nH_{2n}}{2n+1}.\end{aligned}$$

and, by letting $x=1$, we obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^nH_{2n}}{2n+1}=-\frac{\pi\log(2)}{8}.$$

2) We have that

$$\int_0^1\frac{\log(1+x^2)}{1+x^2}\,dx=\int_0^1\sum_{k=0}^{\infty}\frac{(-1)^{k-1}x^{2k}}{k}\sum_{k=0}^{\infty}(-1)^kx^{2k}\,dx=-\int_0^1\sum_{n=1}^{\infty}x^{2n}(-1)^nH_n\,dx=-\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{2n+1}.$$

On the other hand, by letting $x=1/t$, we get

$$\begin{aligned}\int_0^1\frac{\log(1+x^2)}{1+x^2}\,dx&=\int_1^{+\infty}\frac{\log(1+1/t^2)}{1+t^2}\,dt=\int_1^{+\infty}\frac{\log(1+t^2)}{1+t^2}\,dt-2\int_1^{+\infty}\frac{\log(t)}{1+t^2}\,dt\\&=\frac{1}{2}\int_0^{+\infty}\frac{\log(1+t^2)}{1+t^2}\,dt-\int_1^{+\infty}\frac{\log(t)}{1+t^2}\,dt=\frac{\pi\log(2)}{2}-G\end{aligned}$$

because

$$\int_0^{+\infty}\frac{\log(1+t^2)}{1+t^2}\,dt=-2\int_0^{\pi/2}\log(\cos(\theta))\,d\theta=\pi\log(2),$$

where the last step follows from

$$\int_0^{\pi/2}\log(\cos(\theta))\,d\theta=\int_0^{\pi/2}\log(\sin(\theta))\,dt=\frac{1}{2}\int_0^{\pi/2}\log(\sin(2\theta)/2)\,d\theta=\frac{1}{2}\int_0^{\pi/2}\log(\sin(\theta))\,dt-\frac{\pi}{4}\log(2),$$

and

$$\begin{aligned}\int_1^{\infty}\frac{\log(t)}{1+t^2}\,dt&=-\int_0^1\frac{\log(x)}{1+x^2}\,dx=-\int_0^1\log(x)\,d(\arctan(x))=[-\log(x)\arctan(x)]_0^1+\int_0^1\frac{\arctan(x)}{x}\,dx\\&=0+\int_0^1\sum_{k=0}^{\infty}\frac{(-1)^kx^{2k}}{2k+1}\,dx=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}=G.\end{aligned}$$

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