Problem 12253
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Let ABC be a triangle, and let D and E be the contact points of the incircle of ABC with the segments BC and CA, respectively. Let M be the intersection of the line DE and the line through A parallel to BC. Prove that the bisector of $\angle ABC$ passes through the midpoint of DM.

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Solution. We show that the property holds also when in the statement we replace the incircle of ABC and the bisector of $\angle ABC$ with the excircle tangent to the side BC and the bisector of the external angle $\angle ABC$ respectively.

Without loss of generality we may assume that the side BC is along the line $y = 0$, with $B = (b, 0)$, $D = (0, 0)$, $C = (c, 0)$ such that $b < 0 < c$ and $bc \neq -1$. Let $\mathcal{C}$ be the circle of center at $O = (0, 1)$ with radius 1. Note that if $bc < -1$ then $\mathcal{C}$ is the incircle of the triangle ABC. Otherwise, when $-1 < bc < 0$, $\mathcal{C}$ is the excircle tangent to the side BC.
Then the line DE, that is the polar line of the point C with respect to $\mathcal{C}$, is $y = cx$. The circle $\mathcal{C}$, with equation $x^2 + (y - 1)^2 = 1$, intersects the polar line $y = cx$ also at the other point of tangency $E = (\frac{2c}{1+c^2}, \frac{2c^2}{1+c^2})$. Hence the side AC is along the line $(1 - c^2)y = 2c(x - c)$. In a similar way, the side AB is along the line $(1 - b^2)y = 2b(x - b)$. The intersection between the line AB and the line AC is the point $A = (\frac{b+c}{bc+1}, \frac{2bc}{bc+1})$. The line through A parallel to BC is $y = \frac{2bc}{bc+1}$. It follows the intersection of the line DE and such parallel line is $M = (\frac{2b}{bc+1}, \frac{2bc}{bc+1})$. Therefore the midpoint of DM is $F = (\frac{b}{bc+1}, \frac{bc}{bc+1})$.
Finally, it easy to verify that the line BO, that is $x + by = b$, passes through point F,

$$\frac{b}{bc+1} + \frac{b^2c}{bc+1} = b.$$

If $\mathcal{C}$ is the incircle of the triangle ABC then the line BO is the bisector of the internal angle $\angle ABC$. On the other hand, if $\mathcal{C}$ is the excircle tangent to the side BC then the line BO is the bisector of the external angle $\angle ABC$. □