Problem 12252
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Let k, q, and n be positive integers with $k \geq 2$. Prove

$$\sum_{\substack{0<p<k^n \\ k\nmid p}} \left\lceil \frac{\lfloor n-\log_k(p)\rfloor}{q} \right\rceil = \left\lfloor \frac{k^{q-1}(k^{n-1}-1)(k-1)}{k^q-1} \right\rfloor + 1.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $n-1 = aq+r$ with $a, r \in \mathbb{N}$ such that $0 \leq r < q$, then $\lceil \frac{n}{q} \rceil = \lfloor \frac{n-1}{q} \rfloor + 1 = a+1$ and

$$\begin{aligned}
\sum_{\substack{0<p<k^n \\ k\nmid p}} \left\lceil \frac{\lfloor n-\log_k(p)\rfloor}{q} \right\rceil &= \sum_{\substack{0<p<k^n \\ k\nmid p}} \left\lceil \frac{n-\lceil\log_k(p)\rceil}{q} \right\rceil \\
&= \left\lceil \frac{n}{q} \right\rceil + \sum_{j=1}^{n} \sum_{\substack{k^{j-1}<p<k^j \\ k\nmid p}} \left\lceil \frac{n-j}{q} \right\rceil \\
&= \left\lceil \frac{n}{q} \right\rceil + (k-2)\left\lceil \frac{n-1}{q} \right\rceil + \sum_{j=2}^{n} (k-1)^2 k^{j-2} \left\lceil \frac{n-j}{q} \right\rceil \\
&= \left\lceil \frac{n}{q} \right\rceil - \frac{1}{k}\left\lceil \frac{n-1}{q} \right\rceil + (k-1)^2 k^{n-2} \sum_{j=0}^{n-1} \left\lceil \frac{j}{q} \right\rceil \frac{1}{k^j} \\
&= \frac{1}{k}\left(\left\lceil \frac{n}{q} \right\rceil - \left\lceil \frac{n-1}{q} \right\rceil \right) + \frac{k^{q-1+n-1-\lceil \frac{n}{q}\rceil q}(k^{\lceil \frac{n}{q}\rceil q}-1)(k-1)}{k^q-1} \\
&= \frac{k^{r-1}(k^{(a+1)q}-1)(k-1)}{k^q-1} + \begin{cases} \frac{1}{k} & \text{if } r=0 \\ 0 & \text{otherwise} \end{cases} \qquad (1)
\end{aligned}$$

where we applied the following formula with $x = 1/k$ which is obtained by summing by parts,

$$\begin{aligned}
\sum_{j=0}^{n-1} \left\lceil \frac{j}{q} \right\rceil x^j &= \frac{1}{x-1} \sum_{j=0}^{n-1} \left\lceil \frac{j}{q} \right\rceil (x^{j+1}-x^j) = \frac{1}{x-1}\left[\left\lceil \frac{j}{q} \right\rceil x^j \right]_0^n - \frac{1}{x-1} \sum_{j=0}^{n-1} x^{j+1} \left(\left\lceil \frac{j+1}{q} \right\rceil - \left\lceil \frac{j}{q} \right\rceil \right) \\
&= \frac{x^n}{x-1}\left\lceil \frac{n}{q} \right\rceil - \frac{x}{x-1} \sum_{m=0}^{\lfloor \frac{n-1}{q} \rfloor} x^{mq} = \frac{x^n}{x-1}\left\lceil \frac{n}{q} \right\rceil - \frac{x(x^{\lceil \frac{n}{q}\rceil q}-1)}{(x-1)(x^q-1)}.
\end{aligned}$$

On the other hand, the right-hand side of the claimed identity is

$$\begin{aligned}
\left\lfloor \frac{k^{q-1}(k^{aq+r}-1)(k-1)}{k^q-1} \right\rfloor + 1 &= \left\lfloor \frac{k^{r-1}(k^{(a+1)q}-1)(k-1)}{k^q-1} - 1 + \frac{1}{k} \right\rfloor + 1 \\
&= \left\lfloor \frac{k^{r-1}(k^{(a+1)q}-1)(k-1)}{k^q-1} + \frac{1}{k} \right\rfloor
\end{aligned}$$

which is equal to the right-hand side of (1) because k^q-1 divides $k^{(a+1)q}-1$. □