Problem 12251
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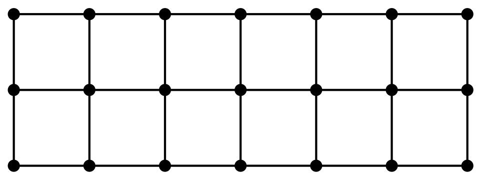
Proposed by R. Tauraso (Italy).

Each point in the plane is colored either red or blue. Show that for any positive real number S, there is a proper convex pentagon of area S all five of whose vertices have the same color. (By a proper convex pentagon we mean a convex pentagon whose internal angles are less than π.)

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Consider a 2×6 square lattice where each 1×1 square has side $l = \frac{\sqrt{S}}{6}$.

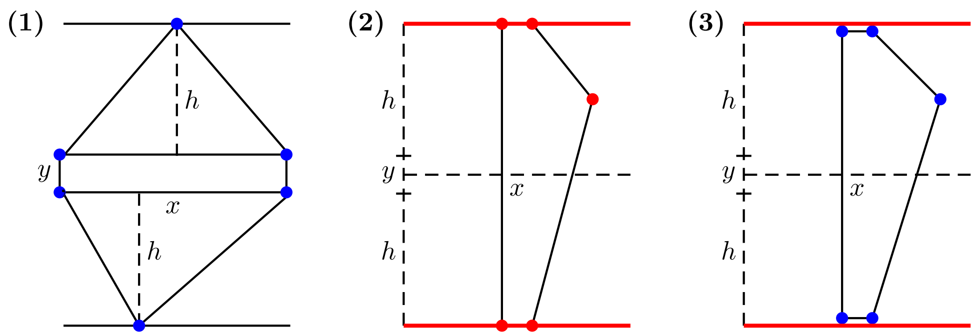


By the Pigeonhole Principle, out of the seven vertices in the first row, at least four have to be of the same color, say, blue. Now we look just at the corresponding four columns. If the second or the third row has at least two blue vertices, we have a rectangle with all blue vertices. Otherwise, the second and the third row have at least three red vertices each and we find a rectangle with all red vertices.
Therefore, after this first step, we have a monochromatic rectangle of sides x and y where $x \in \{l, 2l, 3l, 4l, 5l, 6l\}$ and $y \in \{l, 2l\}$, of area $xy \leq 6l \cdot 2l = \frac{S}{3}$. Without loss of generality, we may assume that the four vertices of monochromatic rectangle are all blue.
We draw two open horizontal segments of lenght x at distance $h = \frac{2(S-xy)}{x} > 0$ from the two horizontal sides of the blue rectangle. We have three cases.

(1) There is at least a blue point in the top open segment or in the bottom open segment.

(2) The points of the two open segments are all red, and in the open rectangle $x \times (y + 2h)$ there is at least a red point.

(3) The points of the two open segments are all red, and all the points of the open rectangle $x \times (y + 2h)$ are blue.



In case **(1)**, we find a proper convex pentagon of area $xy + \frac{xh}{2} = S$ with all blue vertices. Since the area of the rectangle $x \times (y + 2h)$ is

$$x(y + 2h) = xy + 4(S - xy) = 4S - 3xy \geq 4S - S = 3S > 2S,$$

there exists a proper convex pentagon of area S with all red vertices in case **(2)** (two couples of points along the red horizontal segments and one inside the open rectangle), and with all blue vertices in case **(3)** (all five points inside the open rectangle).

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