Problem 12250
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Proposed by D. Marghidanu (Romania).

With $n \geq 4$, let $a_1, \dots, a_n$ be the lengths of the sides of a polygon. Prove

$$\sum_{k=1}^{n} \sqrt{\frac{a_k}{a_1 + \cdots - a_k + \cdots + a_n}} > \frac{2n}{n-1}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let

$$f(x) = \sqrt{\frac{x}{1-2x}} \quad \text{and} \quad g(x) = \frac{2x}{1-x},$$

then, $f(1/3) = g(1/3)$ and $f(x) > g(x)$ for any $x \in (0, 1/2) \setminus \{1/3\}$ because

$$f^2(x) - g^2(x) = \frac{x}{1-2x} - \frac{4x^2}{(1-x)^2} = \frac{x(1-3x)^2}{(1-2x)(1-x)^2}.$$

For $n \geq 4$, by letting $s = a_1 + a_2 + \cdots + a_n$, we have that $\frac{a_1}{s}, \dots, \frac{a_n}{s} \in (0, 1/2)$ cannot be all equal to $1/3$, otherwise we find the contradiction

$$1 = \sum_{k=1}^{n} \frac{a_k}{s} = \frac{n}{3} \geq \frac{4}{3} > 1.$$

Hence

$$\begin{aligned}
\sum_{k=1}^{n} \sqrt{\frac{a_k}{a_1 + \cdots - a_k + \cdots + a_n}} &= \sum_{k=1}^{n} f\left(\frac{a_k}{s}\right) \\
&> \sum_{k=1}^{n} g\left(\frac{a_k}{s}\right) && \frac{a_1}{s}, \dots, \frac{a_n}{s} \text{ are not all equal to } \frac{1}{3} \\
&\geq ng\left(\frac{1}{n}\sum_{k=1}^{n} \frac{a_k}{s}\right) && g \text{ is convex in } (0, 1/2) \\
&= ng\left(\frac{1}{n}\right) = n\frac{2/n}{1-1/n} = \frac{2n}{n-1}.
\end{aligned}$$

□