Problem 12249
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Proposed by F. Stanescu (Romania).

Prove

$$\sum_{k=\lfloor n/2\rfloor}^{n-1}\sum_{m=1}^{n-k}(-1)^{m-1}\frac{k+m}{k+1}\binom{k+1}{m-1}2^{k-m}=\frac{n}{2}$$

for any positive integer n.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. For $k,m\in\mathbb{N}$, let

$$a(k,m)=(-1)^m\binom{k}{m-2}2^{k-m+1}$$

which is zero when $m-2<0$ or $m-2>k$. Then

$$a(k,m+1)-a(k,m)=(-1)^{m-1}2^{k-m}\left(\binom{k}{m-1}+2\binom{k}{m-2}\right)=(-1)^{m-1}\frac{k+m}{k+1}\binom{k+1}{m-1}2^{k-m}.$$

Hence

$$\begin{aligned}\sum_{k=\lfloor n/2\rfloor}^{n-1}\sum_{m=1}^{n-k}(-1)^{m-1}\frac{k+m}{k+1}\binom{k+1}{m-1}2^{k-m}&=\sum_{k=\lfloor n/2\rfloor}^{n-1}\sum_{m=1}^{n-k}(a(k,m+1)-a(k,m))\\&=\sum_{k=\lfloor n/2\rfloor}^{n-1}(a(k,n-k+1)-a(k,1))\\&=\sum_{k=\lfloor n/2\rfloor}^{n-1}(-1)^{n-k+1}\binom{k}{n-k-1}2^{2k-n}\\&=\frac{1}{2}\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-1)^k\binom{n-1-k}{k}2^{n-1-2k}=\frac{n}{2}\end{aligned}$$

where at the last step we applied

$$\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-1)^k\binom{n-1-k}{k}2^{n-1-2k}=n,$$

which is proved below by a generating function argument,

$$\begin{aligned}\sum_{n=1}^{\infty}x^n\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-1)^k\binom{n-1-k}{k}2^{n-1-2k}&=\sum_{k=0}^{\infty}(-1)^k\sum_{n=2k+1}^{\infty}x^n\binom{n-1-k}{k}2^{n-1-2k}\\&=\sum_{k=0}^{\infty}(-1)^k x^{2k+1}\sum_{n=0}^{\infty}\binom{n+k}{k}(2x)^n\\&=\sum_{k=0}^{\infty}(-1)^k x^{2k+1}\cdot\frac{1}{(1-2x)^{k+1}}\\&=\frac{x}{1-2x}\sum_{k=0}^{\infty}\left(\frac{-x^2}{1-2x}\right)^k\\&=\frac{x}{1-2x}\cdot\frac{1}{1-\frac{-x^2}{1-2x}}=\frac{x}{(1-x)^2}=\sum_{n=1}^{\infty}nx^n.\end{aligned}$$

□