

Problem 12248

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Let n be a positive integer, and let x_k be a real number for $1 \leq k \leq 2n$. Let C_n be the $2n$ -by- $2n$ skew-symmetric matrix with (i, j) -entry $\cos(x_i - x_j)$ when $1 \leq i < j \leq 2n$. Prove

$$\det(C_n) = \cos^2(x_1 - x_2 + x_3 - x_4 + \cdots + x_{2n-1} - x_{2n}).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. It is known that, for any skew-symmetric matrix M , $\det(M) = \text{pf}^2(M)$ where $\text{pf}(M)$ is the Pfaffian of that matrix M . Hence it suffices to prove that

$$\text{pf}(C_n) = \cos(x_1 - x_2 + x_3 - x_4 + \cdots + x_{2n-1} - x_{2n}).$$

We show the above property by induction. It holds for $n = 1$:

$$\text{pf}(C_1) = \text{pf} \left(\begin{bmatrix} 0 & \cos(x_1 - x_2) \\ -\cos(x_1 - x_2) & 0 \end{bmatrix} \right) = \cos(x_1 - x_2).$$

Given $n > 1$, we assume that the claim holds for any positive integer less than n . For $2 \leq j \leq 2n$, let $(C_n)_{1,j}$ be the matrix with both the first and the k -th rows and columns removed, then, by the induction hypothesis,

$$\text{pf}((C_n)_{1,j}) = \cos \left(\sum_{k=2}^{j-1} (-1)^k x_k - \sum_{k=j+1}^{2n} (-1)^k x_k \right).$$

Hence, by using the identity $\cos(\alpha) \cos(\beta) = \frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta))$ and by letting $x_{2n+1} = x_1$, we find

$$\begin{aligned} \text{pf}(C_n) &= \sum_{j=2}^{2n} (-1)^j \cos(x_1 - x_j) \text{pf}((C_n)_{1,j}) \\ &= \sum_{j=1}^n \cos(x_1 - x_{2j}) \cos \left(\sum_{k=2}^{2j-1} (-1)^k x_k - \sum_{k=2j+1}^{2n} (-1)^k x_k \right) \\ &\quad - \sum_{j=2}^n \cos(x_1 - x_{2j-1}) \cos \left(\sum_{k=2}^{2j-2} (-1)^k x_k - \sum_{k=2j}^{2n} (-1)^k x_k \right) \\ &= \frac{1}{2} \sum_{j=1}^n \cos \left(\sum_{k=2}^{2j-1} (-1)^k x_k - \sum_{k=2j}^{2n+1} (-1)^k x_k \right) + \frac{1}{2} \sum_{j=1}^n \cos \left(\sum_{k=1}^{2j} (-1)^k x_k - \sum_{k=2j+1}^{2n} (-1)^k x_k \right) \\ &\quad - \frac{1}{2} \sum_{j=2}^n \cos \left(\sum_{k=2}^{2j-1} (-1)^k x_k - \sum_{k=2j}^{2n+1} (-1)^k x_k \right) - \frac{1}{2} \sum_{j=2}^n \cos \left(\sum_{k=1}^{2j-2} (-1)^k x_k - \sum_{k=2j-1}^{2n} (-1)^k x_k \right) \\ &= \frac{1}{2} \sum_{j=1}^n \cos \left(\sum_{k=2}^{2j-1} (-1)^k x_k - \sum_{k=2j}^{2n+1} (-1)^k x_k \right) + \frac{1}{2} \sum_{j=1}^n \cos \left(\sum_{k=1}^{2j} (-1)^k x_k - \sum_{k=2j+1}^{2n} (-1)^k x_k \right) \\ &\quad - \frac{1}{2} \sum_{j=2}^n \cos \left(\sum_{k=2}^{2j-1} (-1)^k x_k - \sum_{k=2j}^{2n+1} (-1)^k x_k \right) - \frac{1}{2} \sum_{j=1}^{n-1} \cos \left(\sum_{k=1}^{2j} (-1)^k x_k - \sum_{k=2j+1}^{2n} (-1)^k x_k \right) \\ &= \frac{1}{2} \cos \left(- \sum_{k=2}^{2n+1} (-1)^k x_k \right) + \frac{1}{2} \cos \left(\sum_{k=1}^{2n} (-1)^k x_k \right) = \cos \left(\sum_{k=1}^{2n} (-1)^k x_k \right). \end{aligned}$$

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