Problem 12248
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Proposed by A. Dzhumadil'daev (Kazakhstan).

Let n be a positive integer, and let x_k be a real number for $1 \leq k \leq 2n$. Let C_n be the $2n$-by-$2n$ skew-symmetric matrix with (i,j)-entry $\cos(x_i - x_j)$ when $1 \leq i < j \leq 2n$. Prove

$$\det(C_n) = \cos^2(x_1 - x_2 + x_3 - x_4 + \cdots + x_{2n-1} - x_{2n}).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. It is known that, for any skew-symmetric matrix M, $\det(M) = \mathrm{pf}^2(M)$ where $\mathrm{pf}(M)$ is the Pfaffian of the matrix M. Hence it suffices to prove that

$$\mathrm{pf}(C_n) = \cos(x_1 - x_2 + x_3 - x_4 + \cdots + x_{2n-1} - x_{2n}).$$

We show the above property by induction. It holds for $n = 1$:

$$\mathrm{pf}(C_1) = \mathrm{pf}\left(\begin{bmatrix} 0 & \cos(x_1 - x_2) \\ -\cos(x_1 - x_2) & 0 \end{bmatrix}\right) = \cos(x_1 - x_2).$$

Given $n > 1$, we assume that the claim holds for any positive integer less than n. For $2 \leq j \leq 2n$, let $(C_n)_{1,j}$ be the matrix with both the first and the j-th rows and columns removed, then, by the induction hypothesis,

$$\mathrm{pf}((C_n)_{1,j}) = \cos\left(\sum_{k=2}^{j-1}(-1)^k x_k - \sum_{k=j+1}^{2n}(-1)^k x_k\right).$$

Hence, by using the identity $\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))$ and by letting $x_{2n+1} = x_1$, we find

$$\begin{aligned}
\mathrm{pf}(C_n) &= \sum_{j=2}^{2n}(-1)^j \cos(x_1 - x_j)\,\mathrm{pf}((C_n)_{1,j}) \\
&= \sum_{j=1}^{n} \cos(x_1 - x_{2j}) \cos\left(\sum_{k=2}^{2j-1}(-1)^k x_k - \sum_{k=2j+1}^{2n}(-1)^k x_k\right) \\
&\quad - \sum_{j=2}^{n} \cos(x_1 - x_{2j-1}) \cos\left(\sum_{k=2}^{2j-2}(-1)^k x_k - \sum_{k=2j}^{2n}(-1)^k x_k\right) \\
&= \frac{1}{2}\sum_{j=1}^{n} \cos\left(\sum_{k=2}^{2j-1}(-1)^k x_k - \sum_{k=2j}^{2n+1}(-1)^k x_k\right) + \frac{1}{2}\sum_{j=1}^{n} \cos\left(\sum_{k=1}^{2j}(-1)^k x_k - \sum_{k=2j+1}^{2n}(-1)^k x_k\right) \\
&\quad - \frac{1}{2}\sum_{j=2}^{n} \cos\left(\sum_{k=2}^{2j-1}(-1)^k x_k - \sum_{k=2j}^{2n+1}(-1)^k x_k\right) - \frac{1}{2}\sum_{j=2}^{n} \cos\left(\sum_{k=1}^{2j-2}(-1)^k x_k - \sum_{k=2j-1}^{2n}(-1)^k x_k\right) \\
&= \frac{1}{2}\sum_{j=1}^{n} \cos\left(\sum_{k=2}^{2j-1}(-1)^k x_k - \sum_{k=2j}^{2n+1}(-1)^k x_k\right) + \frac{1}{2}\sum_{j=1}^{n} \cos\left(\sum_{k=1}^{2j}(-1)^k x_k - \sum_{k=2j+1}^{2n}(-1)^k x_k\right) \\
&\quad - \frac{1}{2}\sum_{j=2}^{n} \cos\left(\sum_{k=2}^{2j-1}(-1)^k x_k - \sum_{k=2j}^{2n+1}(-1)^k x_k\right) - \frac{1}{2}\sum_{j=1}^{n-1} \cos\left(\sum_{k=1}^{2j}(-1)^k x_k - \sum_{k=2j+1}^{2n}(-1)^k x_k\right) \\
&= \frac{1}{2}\cos\left(-\sum_{k=2}^{2n+1}(-1)^k x_k\right) + \frac{1}{2}\cos\left(\sum_{k=1}^{2n}(-1)^k x_k\right) = \cos\left(\sum_{k=1}^{2n}(-1)^k x_k\right).
\end{aligned}$$

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