Problem 12247
(American Mathematical Monthly, Vol.128, April 2021)

Proposed by P. K. Reddy (India).

For positive real constants a, b, and c, prove

$$\int_0^{\pi}\int_0^{\infty}\frac{a}{\pi(x^2+a^2)^{3/2}}\,\frac{x}{\sqrt{x^2+b^2+c^2-2cx\cos(\theta)}}\,dxd\theta=\frac{1}{\sqrt{(a+b)^2+c^2}}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let

$$f(x,y)=\frac{1}{(x^2+y^2+a^2)^{3/2}}\quad\text{and}\quad g(x,y)=\frac{1}{(x^2+y^2+b^2)^{1/2}}.$$

Then the given integral I is the two-dimensional convolution of f and g that we are going to evaluate by using the two-dimensional Fourier transform,

$$I=\frac{a}{2\pi}\iint_{\mathbb{R}^2}f(x,y)g(x-c,y)\,dxdy=\frac{a}{2\pi}(f*g)(c,0)=\frac{a}{2\pi}\mathcal{F}^{-1}(\mathcal{F}(f)\cdot\mathcal{F}(g))(c,0).$$

Let $x=\rho\cos(\theta)$, $y=\rho\sin(\theta)$, $u=r\cos(\varphi)$, $v=r\sin(\varphi)$, then

$$\begin{aligned}\mathcal{F}(f)(u,v)&=\iint_{\mathbb{R}^2}f(x,y)e^{-2\pi i(xu+yv)}\,dxdy=\int_0^{\infty}\frac{\rho}{(\rho^2+a^2)^{3/2}}\left(\int_0^{2\pi}e^{-2\pi i\rho\cos(\theta-\varphi)}d\theta\right)d\rho\\&=\int_0^{\infty}\frac{\rho}{(\rho^2+a^2)^{3/2}}\left(\int_0^{2\pi}e^{-2\pi i\rho\cos(\theta)}d\theta\right)d\rho=\iint_{\mathbb{R}^2}f(x,y)e^{-2\pi irx}\,dxdy\\&=\int_{\mathbb{R}}e^{-2\pi irx}\left(\int_{\mathbb{R}}\frac{dy}{(x^2+y^2+a^2)^{3/2}}\right)dx=\int_{\mathbb{R}}\frac{2e^{-2\pi irx}}{x^2+a^2}\,dx=\frac{2\pi e^{-2\pi ra}}{a}.\end{aligned}$$

Similarly

$$\begin{aligned}\mathcal{F}(g)(u,v)&=\iint_{\mathbb{R}^2}g(x,y)e^{-2\pi i(xu+yv)}\,dxdy=\int_0^{\infty}\frac{\rho}{(\rho^2+b^2)^{1/2}}\left(\int_0^{2\pi}e^{-2\pi i\rho\cos(\theta-\varphi)}d\theta\right)d\rho\\&=\int_0^{\infty}\frac{\rho}{(\rho^2+b^2)^{1/2}}\left(\int_0^{2\pi}e^{-2\pi i\rho\cos(\theta)}d\theta\right)d\rho=\iint_{\mathbb{R}^2}g(x,y)e^{-2\pi irx}\,dxdy\\&=\frac{i}{2\pi r}\iint_{\mathbb{R}^2}g(x,y)\frac{\partial(e^{-2\pi irx})}{\partial x}\,dxdy=-\frac{i}{2\pi r}\iint_{\mathbb{R}^2}\frac{\partial(g(x,y))}{\partial x}e^{-2\pi irx}\,dxdy\\&=\frac{i}{2\pi r}\int_{\mathbb{R}}e^{-2\pi irx}x\left(\int_{\mathbb{R}}\frac{dy}{(x^2+y^2+b^2)^{3/2}}\right)dx=\frac{i}{2\pi r}\int_{\mathbb{R}}\frac{2xe^{-2\pi irx}}{x^2+b^2}\,dx=\frac{e^{-2\pi rb}}{r}.\end{aligned}$$

Finally

$$\begin{aligned}I&=\frac{a}{2\pi}\mathcal{F}^{-1}(\mathcal{F}(f)\cdot\mathcal{F}(g))(c,0)=\iint_{\mathbb{R}^2}\frac{e^{-2\pi ira}}{a}\cdot\frac{e^{-2\pi irb}}{r}\cdot e^{2\pi i(cu+0v)}\,dudv\\&=\int_0^{2\pi}\left(\int_0^{\infty}e^{-2\pi r((a+b)-ic\cos(\varphi))}\,dr\right)d\varphi=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{a+b-ic\cos(\varphi)}\\&=\frac{1}{\sqrt{(a+b)^2+c^2}}.\end{aligned}$$

□