Problem 12246
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Proposed by S. Stewart (Australia).

Prove

$$\sum_{n=2}^{\infty}\frac{\zeta(n)}{n^2}+\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)H_n}{n}=\frac{\pi^2}{6}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that

$$\sum_{n=2}^{\infty}\frac{\zeta(n)}{n^2}=\sum_{k=1}^{\infty}\sum_{n=2}^{\infty}\frac{(1/k^n)}{n^2}=\sum_{k=1}^{\infty}\left(\operatorname{Li}_2\left(\frac{1}{k}\right)-\frac{1}{k}\right).$$

Moreover, for $x\in[0,1)$,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n x^n}{n}&=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{k}\right)\int_0^x t^{n-1}\,dt=\sum_{k=1}^{\infty}\frac{1}{k}\int_0^x\sum_{n=k}^{\infty}t^{n-1}\,dt\\&=\sum_{k=1}^{\infty}\frac{1}{k}\int_0^x\frac{t^{k-1}}{1-t}\,dt=\sum_{k=1}^{\infty}\frac{1}{k}\int_0^x\left(\frac{t^k}{1-t}+t^{k-1}\right)dt\\&=\int_0^x\frac{1}{1-t}\sum_{k=1}^{\infty}\frac{t^k}{k}\,dt+\sum_{k=1}^{\infty}\int_0^x\frac{t^{k-1}}{k}\,dt\\&=-\int_0^x\frac{\ln(1-t)}{1-t}\,dt+\sum_{k=1}^{\infty}\frac{x^k}{k^2}\\&=\frac{\ln^2(1-x)}{2}+\operatorname{Li}_2(x)=-\operatorname{Li}_2\left(\frac{x}{x-1}\right)\end{aligned}$$

where at the last step we applied the Landen's identity. Hence

$$\begin{aligned}\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)H_n}{n}&=\sum_{k=1}^{\infty}\sum_{n=2}^{\infty}\frac{H_n(-1/k)^n}{n}=\sum_{k=1}^{\infty}\left(-\operatorname{Li}_2\left(\frac{-1/k}{-1/k-1}\right)-(-1/k)\right)\\&=\sum_{k=1}^{\infty}\left(-\operatorname{Li}_2\left(\frac{1}{k+1}\right)+\frac{1}{k}\right).\end{aligned}$$

Finally

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{\zeta(n)}{n^2}+\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)H_n}{n}&=\sum_{k=1}^{\infty}\left(\operatorname{Li}_2\left(\frac{1}{k}\right)-\frac{1}{k}\right)+\sum_{k=1}^{\infty}\left(-\operatorname{Li}_2\left(\frac{1}{k+1}\right)+\frac{1}{k}\right)\\&=\sum_{k=1}^{\infty}\left(\operatorname{Li}_2\left(\frac{1}{k}\right)-\operatorname{Li}_2\left(\frac{1}{k+1}\right)\right)=\operatorname{Li}_2(1)=\zeta(2)=\frac{\pi^2}{6}.\end{aligned}$$

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