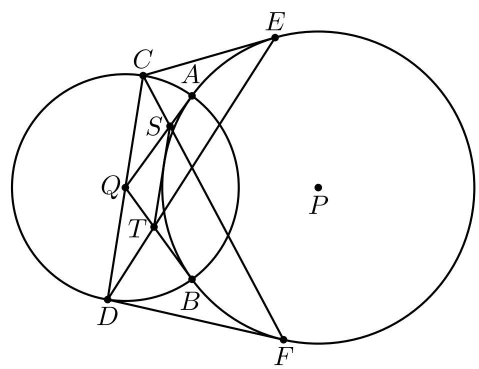
Problem 12245
(American Mathematical Monthly, Vol.128, April 2021)

Proposed by J. Chen (China).

Suppose that two circles α and β, with centers P and Q, respectively, intersect orthogonally at A and B. Let CD be a diameter of β that is exterior to α. Let E and F be points on α such that CE and DF are tangent to α, with C and E on one side of PQ and D and F on the other side of PQ. Let S be the intersection of CF and QA, and let T be the intersection of DE and QB. Prove that ST is parallel to CD and is tangent to α.



Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Without loss of generality we may assume that β is the unit circle centered at $Q=(0,0)$ with $A=(\cos(u),\sin(u))$ and $C=(\cos(v),\sin(v))$ such that $u,v\in(0,\pi)$, $u\neq\pi/2$ and $u\neq v$. Let $B=(\cos(u),-\sin(u))$ and $D=(-\cos(v),-\sin(v))$.

Then $P=(1/\cos(u),0)$ and the radius of α squared is equal to $R^2=|PA|^2=\left(\cos(u)-\frac{1}{\cos(u)}\right)^2+\sin^2(u)=\tan^2(u)$. The line DE is the polar of C with respect to α:

$$\cos(v)x+\sin(v)y-\frac{x+\cos(v)}{\cos(u)}=R^2-\frac{1}{\cos^2(u)}=-1$$

(it is easy to verify that it passes through D), and the line QB is $\sin(u)x+\cos(u)y=0$. Therefore the intersection of DE and QB is

$$T=\left(\frac{\cos(u)-\cos(v)}{1-\cos(u+v)},-\tan(u)\,\frac{\cos(u)-\cos(v)}{1-\cos(u+v)}\right).$$

Similarly, the intersection of the line CF, the polar of D with respect to α), and QA is

$$S=\left(\frac{\cos(u)+\cos(v)}{1+\cos(u-v)},\tan(u)\,\frac{\cos(u)+\cos(v)}{1+\cos(u-v)}\right).$$

Therefore the line ST has equation

$$\sin(v)x-\cos(v)y+\frac{\sin(u)-\sin(v)}{\cos(u)}=0$$

which implies that ST is parallel to the line CD whose equation is $\sin(v)x-\cos(v)y=0$. Finally, the squared distance of the center P from the line ST is

$$\frac{\left(\sin(v)P_x-\cos(v)P_y+\frac{\sin(u)-\sin(v)}{\cos(u)}\right)^2}{(\sin(v))^2+(-\cos(v))^2}=\tan^2(u)=R^2$$

which proves that the line ST is tangent to the circle α. $\square$