Problem 12243
(American Mathematical Monthly, Vol.128, March 2021)

Proposed by M. L. Glasser (USA).

For $a > 0$, evaluate

$$\int_0^a \frac{t}{\sinh(t)\sqrt{1-\frac{\sinh^2(t)}{\sinh^2(a)}}}\,dt.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $s = \tanh(t)$ and $b = \tanh(a)$ then $s \in [0, b] \subset [0, 1)$, $\sinh(\operatorname{arctanh}(s)) = \frac{s}{\sqrt{1-s^2}}$ and

$$\begin{aligned}
\int_0^a \frac{t}{\sinh(t)\sqrt{1-\frac{\sinh^2(t)}{\sinh^2(a)}}}\,dt &= b\int_0^b \frac{\operatorname{arctanh}(s)}{s\sqrt{b^2-s^2}}\,ds \\
&= b\int_0^b \frac{\sum_{k=0}^{\infty}\frac{s^{2k+1}}{2k+1}}{s\sqrt{b^2-s^2}}\,ds \\
&= b\sum_{k=0}^{\infty}\frac{1}{2k+1}\int_0^b \frac{s^{2k}}{\sqrt{b^2-s^2}}\,ds \\
&= \sum_{k=0}^{\infty}\frac{b^{2k+1}}{2k+1}\int_0^{\pi/2}\sin^{2k}(r)\,dr \\
&= \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{1}{4^k}\binom{2k}{k}\frac{b^{2k+1}}{2k+1} \\
&= \frac{\pi}{2}\arcsin(b) = \frac{\pi}{2}\arcsin(\tanh(a))
\end{aligned}$$

where $s = b\sin(r)$ and we applied the Wallis integral. □