Problem 12242
(American Mathematical Monthly, Vol.128, March 2021)

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For $n \geq 1$, let

$$I_n = \int_0^1 \frac{\left(\sum_{k=0}^{n} x^k/(2k+1)\right)^{2022}}{\left(\sum_{k=0}^{n+1} x^k/(2k+1)\right)^{2021}}\,dx.$$

Let $L = \lim_{n\to\infty} I_n$. Compute L and $\lim_{n\to\infty} n(I_n - L)$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We show a more general result: *If $(a_n)_{n\geq 0}$ is a sequence of non-negative real numbers such that $\lim_{n\to\infty} na_n = l$ then, for any $m \in \mathbb{N}$,*

$$\lim_{n\to\infty} I_n(m) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} = L \qquad \textit{and} \qquad \lim_{n\to\infty} n(I_n(m) - L) = -\lim_{n\to\infty} na_n = -l$$

where $S_n(x) = \sum_{k=0}^{n} a_k x^k$ and $I_n(m) = \int_0^1 \frac{(S_n(x))^{m+1}}{(S_{n+1}(x))^m}\,dx$.
Hence, in the our case, we have $a_n = \frac{1}{2n+1}$ and it follows that

$$L = \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)} = 2\sum_{k=0}^{\infty}\left(\frac{1}{2k+1} - \frac{1}{2k+2}\right) = 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k} = 2\ln(2),$$

and

$$\lim_{n\to\infty} n(I_n - L) = -\lim_{n\to\infty}\frac{n}{2n+1} = -\frac{1}{2}.$$

Proof of the general result. We note that for $x \in [0,1]$,

$$\frac{(S_n(x))^{m+1}}{(S_{n+1}(x))^m} = \Big(1 - \underbrace{\frac{a_{n+1}x^{n+1}}{S_{n+1}(x)}}_{\in[0,1]}\Big)^m S_n(x) \geq \left(1 - m\frac{a_{n+1}x^{n+1}}{S_{n+1}(x)}\right) S_n(x)$$

which implies

$$0 \leq S_n(x) - \frac{(S_n(x))^{m+1}}{(S_{n+1}(x))^m} \leq m\frac{a_{n+1}x^{n+1}}{S_{n+1}(x)} S_n(x) \leq m a_{n+1} x^{n+1}.$$

Therefore, as $n \to \infty$,

$$0 \leq \int_0^1 S_n(x)\,dx - I_n(m) \leq m\frac{a_{n+1}}{n+2} \to 0,$$

and we find that

$$\lim_{n\to\infty} I_n(m) = \lim_{n\to\infty}\int_0^1 S_n(x)\,dx = \lim_{n\to\infty}\sum_{k=0}^{n} a_k \int_0^1 x^k\,dx = \sum_{k=0}^{\infty}\frac{a_k}{k+1} = L.$$

Moreover, as $n \to \infty$,

$$n(I_n(m) - L) = -n\left(\int_0^1 S_n(x)\,dx - I_n(m)\right) + n\left(\int_0^1 S_n(x)\,dx - L\right) \to 0 + l = l$$

because $a_n \to 0$,

$$0 \leq n\left(\int_0^1 S_n(x)\,dx - I_n(m)\right) \leq nm\frac{a_{n+1}}{n+2} \to 0,$$

and, by StolzCesro theorem,

$$\begin{aligned}\lim_{n\to\infty} n\left(\int_0^1 S_n(x)\,dx - L\right) &= \lim_{n\to\infty}\frac{\int_0^1 (S_{n+1}(x) - S_n(x))\,dx}{\frac{1}{n+1} - \frac{1}{n}} \\ &= \lim_{n\to\infty} -n(n+1)a_{n+1}\int_0^1 x^{n+1}\,dx \\ &= -\lim_{n\to\infty}\frac{n(n+1)a_{n+1}}{n+2} = -l.\end{aligned}$$

□