Problem 12241
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Prove

$$\sum_{n=1}^{\infty}(-1)^n n\left(\frac{1}{4n}-\ln(2)+\sum_{k=n+1}^{2n}\frac{1}{k}\right)=\frac{\ln(2)-1}{8}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. For $N \geq 1$, let

$$S_N=\frac{1}{8}\left((-1)^N-1+2\ln(2)-\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k}+2(-1)^N(2N+1)\left(\sum_{k=1}^{2N}\frac{(-1)^{k-1}}{k}-\ln(2)\right)\right).$$

Then, as $N \to +\infty$,

$$\begin{aligned}\sum_{n=1}^{N}(-1)^n n\left(\frac{1}{4n}-\ln(2)+H_{2n}-H_n\right)&=\sum_{n=1}^{N}(S_n-S_{n-1})=S_N\\&=\frac{1}{8}\left((-1)^N-1+2\ln(2)-\ln(2)+o(1)+2(-1)^N(2N+1)\left(-\frac{1}{4N}+o(1/N)\right)\right)\\&=\frac{\ln(2)-1}{8}+o(1)\to\frac{\ln(2)-1}{8}\end{aligned}$$

where we used the fact that

$$\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k}=\ln(2)-\frac{(-1)^N}{2N}+o(1/N).$$

□