Problem 12239
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Determine all positive integers r such that there exist at least two pairs of positive integers (m, n) satisfying the equation $2^m = n! + r$.

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Solution. Let r be a positive integer such that

$$r = 2^m - n! = 2^M - N!$$

with $M > m$, $N > n$ and $n, m, N, M \in \mathbb{N}^+$. If $N \geq n + 2$ then

$$2^m(2^{M-m} - 1) = 2^M - 2^m = N! - n! = n!\left(\underbrace{\overbrace{(N-n)\dots(n+2)(n+1)}^{\text{even}} - 1}_{\text{odd}}\right)$$

which implies that 2^m divides $n!$. On the other hand $2^m > n!$ and we have a contradiction. Therefore it follows that $N = n + 1$.

If $n \geq 5$ then

$$2^m(\underbrace{2^{M-m} - 1}_{\text{odd}}) = 2^M - 2^m = N! - n! = n!n = n^2(n-1)(n-2)(n-3)!$$

If n is odd then the odd factor n^2 divides $2^{M-m} - 1$. Otherwise, if n is even then the odd factor $(n-1)(n-3)$ divides $2^{M-m} - 1$. On the other hand

$$2^{M-m} - 1 = \frac{N! - n!}{2^m} = \frac{n!n}{2^m} < n$$

and we have a contradiction because n^2 and $(n-1)(n-3)$ are both greater than n. Therefore it follows that $1 \leq n \leq 4$.

By direct computation, we find that for $1 \leq n \leq 4$ and $N = n + 1$:

$$2^m(2^{M-m} - 1) = n!n \in \{1,\ 4 = 2^2 \cdot 1,\ 18 = 2 \cdot 9,\ 96 = 2^5 \cdot 3\}.$$

Hence we have only two possible values for r, i. e. 2 and 8:

$$2 = 2^2 - 2! = 2^3 - 3! \qquad \text{and} \qquad 8 = 2^5 - 4! = 2^7 - 5!.$$

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