Problem 12238
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Let $ABCD$ be a convex quadrilateral with $AD = BC$. Let P be the intersection of the diagonals AC and BD, and let K and L be the circumcenters of triangles PAD and PBC, respectively. Show that the midpoints of segments AB, CD, and KL are collinear.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We may assume without loss of generality that the four vertices of the convex quadrilateral are the following points in the complex plane,

$$A = -z_1, \quad B = z_1, \quad C = a + z_2, \quad D = a - z_2$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, and $a > 0$.
We note that the midpoints of the segments AB and CD are along the real axis:

$$\frac{A+B}{2} = 0 \qquad \text{and} \qquad \frac{C+D}{2} = a.$$

Moreover, since $AD = BC$, we have that

$$|D - A| = |a - (z_2 - z_1)| = |a + z_2 - z_1| = |C - B| \implies \text{Re}(z_2 - z_1) = x_2 - x_1 = 0 \implies x_1 = x_2.$$

Let $x = x_1 = x_2$. The intersection of the diagonals AC and BD is given by

$$P = \frac{a^2 y_1 + 2x^2(y_2 - y_1)}{a(y_1 + y_2)} + i\frac{x(y_2 - y_1)}{a}.$$

By the Lemma given below, $K = C(P, A, D)$ and $L = C(P, B, C)$ and we find that

$$\text{Im}(K) = -\frac{a^2 - 4x^2 + (y_1 + y_2)^2}{4(y_1 + y_2)} \quad \text{and} \quad \text{Im}(L) = \frac{a^2 - 4x^2 + (y_1 + y_2)^2}{4(y_1 + y_2)}.$$

It follows that $\text{Im}\left(\frac{K+L}{2}\right) = 0$ which means that also the midpoint of the segment KL is along the real axis. Hence we may conclude that the midpoints of segments AB, CD, and KL are collinear.
□

Lemma *The circumcenter of a triangle with vertices P_1, P_2, P_3 is given by*

$$C(P_1, P_2, P_3) = i\,\frac{P_1(|P_2|^2 - |P_3|^2) + P_2(|P_3|^2 - |P_1|^2) + P_3(|P_1|^2 - |P_2|^2)}{2\text{Im}(P_1\overline{P_3} + P_2\overline{P_1} + P_3\overline{P_2})}.$$

Proof. Let R be the radius of the circumcircle of P_1, P_2, P_3 then the circumcenter C is defined by the equations $|C - P_1|^2 = |C - P_2|^2 = |C - P_3|^2 = R^2$, and after expanding the squares we find

$$\begin{cases} \overline{P_1}\,C + P_1\,\overline{C} & = |P_1|^2 + C^2 - R^2 \\ \overline{P_2}\,C + P_2\,\overline{C} & = |P_2|^2 + C^2 - R^2 \\ \overline{P_3}\,C + P_3\,\overline{C} & = |P_3|^2 + C^2 - R^2 \end{cases} \Leftrightarrow \begin{cases} \overline{(P_2 - P_1)}\,C + (P_2 - P_1)\,\overline{C} & = |P_2|^2 - |P_1|^2 \\ \overline{(P_3 - P_1)}\,C + (P_3 - P_1)\,\overline{C} & = |P_3|^2 - |P_1|^2 \end{cases}$$

and after eliminating $\overline{C}$, we get

$$C = \frac{(P_3 - P_1)(|P_2|^2 - |P_1|^2) - (P_2 - P_1)(|P_3|^2 - |P_1|^2)}{(P_3 - P_1)\overline{(P_2 - P_1)} - (P_2 - P_1)\overline{(P_3 - P_1)}}$$

which is equivalent to the given formula. □