Problem 12237
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Proposed by D. E. Knuth (USA).

Let $x_0 = 1$ *and* $x_{n+1} = x_n + \lfloor x_n^{3/10} \rfloor$ *for* $n \geq 0$. *What are the first* 40 *decimal digits of* x_n *when* $n = 10^{100}$*?*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We consider the sequence

$$z_n = \left(\frac{7n}{10}\right)^{10/7} \qquad \text{for } n \geq 0.$$

i) We show by induction that $\forall n \geq 6$, $x_n < z_n$.
We have that $x_6 = 7 < z_6 \approx 7.77$ and for $n \geq 6$, if $x_n < z_n$ then

$$x_{n+1} = x_n + \lfloor x_n^{3/10} \rfloor \leq x_n + x_n^{3/10} < z_n + z_n^{3/10} \leq z_{n+1}$$

where the last inequality is equivalent to

$$\left(\frac{7n}{10}\right)^{10/7} + \left(\frac{7n}{10}\right)^{3/7} \leq \left(\frac{7(n+1)}{10}\right)^{10/7}$$

that is

$$1 + \frac{10/7}{n} \leq \left(1 + \frac{1}{n}\right)^{10/7}$$

which holds because

$$1 + (10/7)x \leq (1+x)^{10/7} \qquad \text{for } x \geq 0.$$

ii) We show by induction that $\forall n \geq 500$, $z_n - 2n < x_n$.
We have that $x_{500} = 3857 > z_{500} - 2 \cdot 500 \approx 3309.06$ and for $n \geq 500$, if $x_n > z_n$ then

$$\begin{aligned} x_{n+1} &= x_n + \lfloor x_n^{3/10} \rfloor \geq x_n + x_n^{3/10} - 1 \\ &> (z_n - 2n) + (z_n - 2n)^{3/10} - 1 \geq z_{n+1} - 2(n+1) \end{aligned}$$

where the last inequality holds if and only if

$$\left(\frac{7n}{10}\right)^{10/7} + \left(\left(\frac{7n}{10}\right)^{10/7} - 2n\right)^{3/10} \geq \left(\frac{7(n+1)}{10}\right)^{10/7} - 1$$

that is

$$1 + \frac{10/7}{n}\left(1 - \frac{2a}{n^{3/7}}\right)^{3/10} + \frac{a}{n^{10/7}} \geq \left(1 + \frac{1}{n}\right)^{10/7}$$

where $a = (10/7)^{10/7} \approx 1.66$, which holds because for $x \in [0, 1/500]$,

$$\begin{aligned} 1 + (10/7)x\left(1 - 2ax^{3/7}\right)^{3/10} + ax^{10/7} &\geq 1 + (10/7)x\left(1 - \frac{2ax^{3/7}}{3}\right) + ax^{10/7} \\ &= 1 + (10/7)x + \frac{ax^{10/7}}{21} \geq 1 + (10/7)x + x^2 \geq (1+x)^{10/7}. \end{aligned}$$

For $n = 10^{100}$, the integer part of z_n has 143 digits and

$$\begin{aligned} z_n = \left(7 \cdot 10^{99}\right)^{10/7} = 7^{10/7} 10^{3/7} 10^{141} &= 4323687954442595126321573916177882577073381\ldots 23.27\ldots \\ z_n - 2 \cdot 10^{100} &= 4323687954442595126321573916177882577073379\ldots 23.27\ldots \end{aligned}$$

By i) and ii), it follows that for $n = 10^{100}$, $z_n - 2n < x_n < z_n$ and therefore the first 40 decimal digits of $z_n - 2n$, x_n and z_n are the same:

$$4323687954442595126321573916177882577073.$$

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