Problem 12236
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Let p_k be the k-th prime number, and let $a_n = \prod_{k=1}^{n} p_k$. Prove that for $n \in \mathbb{N}$ every positive integer less than a_n can be expressed as a sum of at most $2n$ distinct divisors of a_n.

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Solution. We show by induction with respect to n that the statement is true.

The base case. For $n = 1$, the divisors of $a_2 = 2$ are 1 and 2. Then for $N = 1$ we just need the divisor 1 and we are done.

The induction step. We assume $n > 1$, and let N be a positive integer less than a_n. We divide N by p_n getting a quotient q and a remainder r such that

$$0 \leq q \leq \frac{N}{p_n} < \frac{a_n}{p_n} = a_{n-1} \qquad \text{and} \qquad 0 \leq r < p_n.$$

We also notice that any squarefree number less than p_n is a divisor of a_{n-1}.
If $q = 0$ then we let $m = 0$. If $q \geq 1$, by the induction hypothesis, there exist $d_1 < d_2 < \cdots < d_m$ distinct divisors of a_{n-1} with $1 \leq m \leq 2(n-1)$ such that $q = d_1 + d_2 + \cdots + d_m$.
Moreover, if $r = 0$ then we let $l = 0$. If $r \geq 1$ is a squarefree number then, by the above remark, it is a divisor of $a_{n-1} = a_n/p_n$ and we let $t_1 = r$ and $l = 1$. Otherwise, if $r \geq 1$ is not a squarefree number then, by the next Lemma, there are two distict squarefree numbers $t_1 < t_2$ such that $r = t_1 + t_2$ and we let $l = 2$.
Hence we may conclude that N can be written as sum of at most $m + l \leq 2(n-1) + 2 = 2n$ distinct divisors of a_n:

$$N = qp_n + r = \sum_{j=1}^{m} d_j p_n + \sum_{j=1}^{l} t_j.$$

□

Lemma *Any non-squarefree positive integer N can be written as the sum of two distinct squarefree positive numbers.*

Proof. Let $(1, 2, 3, 5, 6, \ldots) = (s_j)_{j\geq 1}$ be the strictly increasing sequence of the squarefree numbers and let $S(N)$ be the number of squarefree numbers less or equal than N.
By the inclusion-exclusion principle,

$$\begin{aligned} S(N) &= \sum_{k\leq\lfloor\sqrt{N}\rfloor} \mu(k) \left\lfloor \frac{N}{k^2} \right\rfloor = N\sum_{k=1}^{\infty} \frac{\mu(k)}{k^2} - N \sum_{k\geq\lfloor\sqrt{N}\rfloor+1} \frac{\mu(k)}{k^2} - \sum_{k\leq\lfloor\sqrt{N}\rfloor} \mu(k) \left(\frac{N}{k^2} - \left\lfloor \frac{N}{k^2} \right\rfloor \right) \\ &\geq \frac{6N}{\pi^2} - N \sum_{k\geq\lfloor\sqrt{N}\rfloor+1} \frac{1}{k^2} - \sum_{k\leq\lfloor\sqrt{N}\rfloor} 1 \geq \frac{6N}{\pi^2} - N \int_{\lfloor\sqrt{N}\rfloor}^{\infty} \frac{dx}{x^2} - \lfloor\sqrt{N}\rfloor \\ &\geq \frac{6N}{\pi^2} - \frac{N}{\lfloor\sqrt{N}\rfloor} - \lfloor\sqrt{N}\rfloor \geq \frac{6N}{\pi^2} - 2\sqrt{N} - 1. \end{aligned}$$

We assume now that N, which is not a squarefree number, cannot be written as the sum of two distinct squarefree numbers. It follows that the set

$$\{s_1, s_2, \ldots, s_{S(N)}, N - s_1, N - s_2, \ldots, N - s_{S(N)}\} \subset [1, N-1]$$

contains at least $2S(N) - 1$ distinct integers, and therefore

$$N - 1 \geq 2S(N) - 1 \geq \frac{12N}{\pi^2} - 4\sqrt{N} - 2 - 1 \implies N < 361.7$$

On the other hand, by direct calculation, any non-squarefree integer $1 \leq N \leq 361$ can be written as the sum of two distinct squarefree numbers. Hence we have a contradiction and the proof of the lemma is complete. □