Problem 12233
(American Mathematical Monthly, Vol.128, February 2021)

Proposed by C. R. Pranesachar (India).

Let n and k be positive integers with $1 \leq k < \frac{n+1}{2}$. For $1 \leq r \leq n$, let $h(r)$ be the number of k-element subsets of $\{1, \ldots, n\}$ that do not contain consecutive elements but that do contain r. Prove
(a) $h(r) = h(r+1)$ when $r \in \{k, \ldots, n-k\}$.
(b) $h(k-1) = h(k) \pm 1$.
(c) $h(r) > h(r+2)$ when $r \in \{1, \ldots, k-2\}$ and r is odd.
(d) $h(r) < h(r+2)$ when $r \in \{1, \ldots, k-2\}$ and r is even.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. For $n \geq 1$, let $F_n(x)$ be the n-th *Fibonacci polynomial* defined as

$$F_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} x^k$$

then it is easy to verify that $F_1(x) = F_2(x) = 1$, and $F_{n+1}(x) = F_n(x) + xF_{n-1}(x)$ for $n \geq 2$. Since the number of k-element subsets of $\{1, \ldots, n\}$ that do not contain consecutive elements is equal to $\binom{n+1-k}{k} = [x^k]F_{n+2}(x)$, it follows that the number of such subsets that contain r is equal

$$h(n, k, r) = [x^{k-1}]F_r(x)F_{n+1-r}(x).$$

Notice the following recurrence holds

$$\begin{aligned}
h(n, k, r) - h(n, k, r+1) &= [x^{k-1}](F_r(x)F_{n+1-r}(x) - F_{r+1}(x)F_{n-r}(x)) \\
&= [x^{k-1}](F_r(x)(F_{n-r}(x) + xF_{n-r-1}(x)) - (F_r(x) + xF_{r-1}(x))F_{n-r}(x)) \\
&= [x^{k-2}](F_r(x)F_{n-1-r}(x)) - F_{r-1}(x)F_{n-r}(x)) \\
&= h(n-2, k-1, r) - h(n-2, k-1, r-1).
\end{aligned}$$

Moreover for $1 \leq k < \frac{n+1}{2}$, $h(n, k, 1) = 1$ and

$$h(n, k, 1) = \binom{n-k}{k-1}, \ h(n, k, 2) = \binom{n-1-k}{k-1}, \ h(n, k, 3) = \binom{n-2-k}{k-1} + \binom{n-1-k}{k-2}.$$

We prove the required properties by induction with respect to n. They trivially hold for $n = 2, 3$. As regards the induction step, for $n \geq 4$, we have that:

(a) $h(n, 1, r) = 1$ and for $k \geq 2$ when $r \in \{k, \ldots, n-k\}$,

$$h(n, k, r) - h(n, k, r+1) = -(h(n-2, k-1, r-1) - h(n-2, k-1, r)) = 0;$$

(b) $h(n, 2, 1) - h(n, 2, 2) = (n-2) - (n-3) = 1$ and for $k \geq 3$,

$$h(n, k, k-1) - h(n, k, k) = -(h(n-2, k-1, k-2) - h(n-2, k-1, k-1)) \in \{1, -1\};$$

(c) $h(n, k, 1) - h(n, k, 3) > 0$ and if $r \geq 3$ is odd with $r \in \{1, \ldots, k-2\}$, then $r-1 \geq 2$ is even and by the above recurrence

$$h(n, k, r) - h(n, k, r+2) = -(h(n-2, k-1, r-1) - h(n-2, k-1, r+1)) > 0;$$

(d) if $r \geq 2$ is even with $r \in \{1, \ldots, k-2\}$, then $r-1 \geq 1$ is odd and by the above recurrence

$$h(n, k, r) - h(n, k, r+2) = -(h(n-2, k-1, r-1) - h(n-2, k-1, r+1)) < 0.$$

□