Problem 12232
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Proposed by S. Stewart (Australia).

Prove

$$\int_0^1\int_0^1 \frac{1}{\sqrt{x(1-x)}\sqrt{y(1-y)}\sqrt{1-xy}}\,dx\,dy = \frac{1}{4\pi}\left(\int_0^{+\infty} e^{-t}t^{-3/4}\,dt\right)^4.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Since $\frac{1}{\sqrt{1-xy}} = \sum_{k=0}^{\infty}\binom{2k}{k}\frac{(xy)^k}{4^k}$ for $x, y \in [0,1)$, it follows that

$$\begin{aligned}\int_0^1\int_0^1 \frac{1}{\sqrt{x(1-x)}\sqrt{y(1-y)}\sqrt{1-xy}}\,dx\,dy &= \sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{4^k}\int_0^1\int_0^1 \frac{(xy)^k}{\sqrt{x(1-x)}\sqrt{y(1-y)}}\,dx\,dy\\ &= \sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{4^k}\left(\int_0^1 x^{k-1/2}(1-x)^{-1/2}\,dx\right)^2\\ &= \sum_{k=0}^{\infty}\binom{2k}{k}\frac{B^2(k+1/2,1/2)}{4^k}\\ &= \sum_{k=0}^{\infty}\binom{2k}{k}\frac{\Gamma^2(k+1/2)\Gamma^2(1/2)}{4^k\Gamma^2(k+1)}\\ &= \pi^2\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3}{4^{3k}} = \pi^2\,{}_3F_2\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{matrix};1\right] = 4K^2\left(\frac{1}{2}\right)\\ &= \frac{\Gamma^4\left(\frac{1}{4}\right)}{4\pi} = \frac{1}{4\pi}\left(\int_0^{+\infty} e^{-t}t^{-3/4}\,dt\right)^4\end{aligned}$$

where, for $|k| < 1$,

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

is the complete elliptic integral of the first kind, and it is know that

$$K\left(\frac{1}{2}\right) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}}.$$

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