Problem 12231
(American Mathematical Monthly, Vol.128, February 2021)

Proposed by G. Apostolopoulos (Greece).

For an acute triangle ABC with circumradius R and inradius r, prove

$$\sec\left(\frac{A-B}{2}\right)+\sec\left(\frac{B-C}{2}\right)+\sec\left(\frac{C-A}{2}\right)\leq \frac{R}{r}+1.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let h_a, w_a, m_a, M_a be the height, the angle bisector, the median and the midpoint with respect to the side a. Similar notations are used for the other sides b and c. We notice that

$$\frac{w_a}{h_a}=\frac{2bc\cos\left(\frac{A}{2}\right)}{b+c}\cdot\frac{a}{bc\sin(A)}=\frac{2\cos\left(\frac{A}{2}\right)}{\sin(B)+\sin(C)}=\frac{\cos\left(\frac{A}{2}\right)}{\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}=\sec\left(\frac{B-C}{2}\right).$$

Hence the given inequality is equivalent to

$$\frac{w_a}{h_a}+\frac{w_b}{h_b}+\frac{w_c}{h_c}\leq\frac{R}{r}+1,$$

and since $w_a\leq m_a$, $w_b\leq m_b$, $w_c\leq m_c$, it suffices to show the stronger inequality

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\leq\frac{R}{r}+1.$$

Indeed, we have that

$$\begin{aligned}\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}&=\frac{am_a+bm_b+cm_c}{2S}\\&\leq\frac{a(R+|OM_a|)+b(R+|OM_b|)+c(R+|OM_c|)}{2S}\\&=\frac{(a+b+c)R}{2S}+\frac{a|OM_a|+b|OM_b|+b|OM_c|}{2S}=\frac{R}{r}+1\end{aligned}$$

where S is the area of the triangle, O is the circumcenter, and, after recalling that the triangle is acute, we applied

$$2S=a|OM_a|+b|OM_b|+b|OM_c|.$$

□