**Problem 12229**
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Proposed by M. Omarjee (France).

*Let $f : [0,1] \to \mathbb{R}$ be a function that has a continuous second derivative and that satisfies $f(0) = f(1)$ and $\int_0^1 f(x)\,dx = 0$. Prove*

$$30240\left(\int_0^1 xf(x)\,dx\right)^2 \leq \int_0^1 \left(f''(x)\right)^2\,dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

*Solution.* By integrating by parts twice, we have that

$$\begin{aligned}
12\int_0^1 xf(x)\,dx &= 12\int_0^1 xf(x)\,dx - 6\int_0^1 f(x)\,dx = \int_0^1 (12x-6)f(x)\,dx = \int_0^1 f(x)\,d(6x^2-6x+1)\\
&= \left[(6x^2-6x+1)f(x)\right]_0^1 - \int_0^1 (6x^2-6x+1)f'(x)\,dx\\
&= (f(1)-f(0)) - \int_0^1 (6x^2-6x+1)f'(x)\,dx\\
&= -\int_0^1 f'(x)\,d((2x^3-3x^2+x))\\
&= -\left[(2x^3-3x^2+x)f'(x)\right]_0^1 + \int_0^1 (2x^3-3x^2+x)f''(x)\,dx\\
&= \int_0^1 (2x^3-3x^2+x)f''(x)\,dx.
\end{aligned}$$

Hence, by Cauchy-Schwarz inequality,

$$\left(12\int_0^1 xf(x)\,dx\right)^2 \leq \int_0^1 (2x^3-3x^2+x)^2\,dx \cdot \int_0^1 \left(f''(x)\right)^2\,dx.$$

Finally, since $\int_0^1 (2x^3-3x^2+x)^2\,dx = \frac{1}{210}$, and $12^2 \cdot 210 = 30240$, it follows that

$$30240\left(\int_0^1 xf(x)\,dx\right)^2 \leq \int_0^1 \left(f''(x)\right)^2\,dx.$$

$\square$