Problem 12228
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Proposed by H. Grandmontagne (France).

Prove

$$\int_0^1 \frac{(\ln(x))^2 \ln\left(2\sqrt{x}/(x^2+1)\right)}{x^2-1}\,dx = 2G^2\,,$$

where G is Catalan's constant $\sum_{n=0}^{\infty}(-1)^n/(2n+1)^2$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution.
We consider, for $x > 0$, the *digamma function* $\psi(x)$ and its derivatives of order $m \geq 1$:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+x}\right), \quad \psi^{(m)}(x) = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(n+x)^{m+1}}.$$

We recall some known values of $\psi(x)$ and $\psi^{(m)}(x)$ up to order 3 at $x = 1/2, 1/4, 3/4$:

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\ln(2), \qquad \psi'\left(\frac{1}{2}\right) = \frac{\pi^2}{2}, \qquad \psi''\left(\frac{1}{2}\right) = -14\zeta(3), \qquad \psi'''\left(\frac{1}{2}\right) = \pi^4,$$

$$\psi\left(\frac{1}{4}\right) = -\gamma - 3\ln(2) - \frac{\pi}{2}, \quad \psi'\left(\frac{1}{4}\right) = 8G + \pi^2, \quad \psi''\left(\frac{1}{4}\right) = -56\zeta(3) - 2\pi^3,$$

$$\psi\left(\frac{3}{4}\right) = -\gamma - 3\ln(2) + \frac{\pi}{2}, \quad \psi'\left(\frac{3}{4}\right) = -8G + \pi^2, \quad \psi''\left(\frac{3}{4}\right) = -56\zeta(3) + 2\pi^3.$$

Moreover

$$\psi'''\left(\frac{1}{4}\right) + \psi'''\left(\frac{1}{4}\right) = 16\pi^4.$$

We need also the following two identities. Let $H_k = \sum_{n=1}^{k} \frac{1}{n}$ then, for $x > 0$,

$$S_2(x) := \sum_{k=1}^{\infty} \frac{H_k}{(k+x)^2} = (\gamma + \psi(x))\,\psi'(x) - \frac{\psi''(x)}{2}, \tag{1}$$

and

$$S_3(x) := \sum_{k=1}^{\infty} \frac{H_k}{(k+x)^3} = -\frac{(\psi'(x))^2}{2} - \frac{(\gamma + \psi(x))\,\psi''(x)}{2} + \frac{\psi'''(x)}{4}. \tag{2}$$

Indeed, we have that

$$\begin{aligned}
S_2(x) &= \sum_{k=1}^{\infty} \frac{H_k}{(k+x)^2} = \sum_{k=1}^{\infty} \frac{1}{(k+x)^2} \sum_{n=1}^{k} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=n}^{\infty} \frac{1}{(k+x)^2} = \sum_{n=1}^{\infty} \frac{\psi'(x+n)}{n} \\
&= -\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 \frac{t^{x+n-1}\ln(t)}{1-t}\,dt = -\int_0^1 \frac{t^{x-1}\ln(t)}{1-t} \sum_{n=1}^{\infty} \frac{t^n}{n}\,dt = \int_0^1 \frac{t^{x-1}\ln(t)\ln(1-t)}{1-t}\,dt \\
&= \lim_{y\to 0^+} \frac{\partial^2}{\partial x \partial y} \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \lim_{y\to 0^+} \frac{\partial^2}{\partial x \partial y} \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \\
&= \lim_{y\to 0^+} \frac{\partial}{\partial y}\left(\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}(\psi(x) - \psi(x+y))\right) \\
&= \lim_{y\to 0^+} \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\left((\psi(y) - \psi(x+y))(\psi(x) - \psi(x+y)) - \psi'(x+y)\right) \\
&= \lim_{y\to 0^+} \left(\frac{1}{y} - \gamma + O(y)\right)\Bigg(\left(-\frac{1}{y} - \gamma + O(y) - \psi(x)\right)\left(-\psi'(x)y - \frac{\psi''(x)}{2}y^2 + O(y^3)\right) \\
&\qquad\qquad - (\psi'(x) + \psi''(x)y + O(y^2))\Bigg) \\
&= (\gamma + \psi(x))\,\psi'(x) - \frac{\psi''(x)}{2},
\end{aligned}$$

and the proof of (1) is complete. We obtain (2) by differentiating (1) with respect to x.

The given integral can be written as

$$\int_0^1 \frac{(\ln(x))^2 \ln\left(2\sqrt{x}/(x^2+1)\right)}{x^2-1}\, dx = \ln(2) I_1 + \frac{I_2}{2} - I_3 = 2G^2$$

where at the last step we applied

$$I_1 := \int_0^1 \frac{\ln^2(x)}{x^2-1}\, dx = -\sum_{k=0}^{\infty} \int_0^1 \ln^2(x) x^{2k}\, dx = -2\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} = -\frac{7\zeta(3)}{4},$$
$$I_2 := \int_0^1 \frac{\ln^3(x)}{x^2-1}\, dx = 6\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{16},$$

and, by using (2) with the given evaluations listed at the beginning,

$$\begin{aligned}
I_3 &:= \int_0^1 \frac{\ln^2(x)\ln(x^2+1)}{x^2-1}\, dx = 2\sum_{k=0}^{\infty}\int_0^1 \ln^2(x) x^{2k} \sum_{n=1}^{k} \frac{(-1)^n}{n}\, dx = 2\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \sum_{n=1}^{k} \frac{(-1)^n}{n} \\
&= -2\sum_{k=1}^{\infty} \frac{H_k}{(2k+1)^3} + 2\sum_{k=1}^{\infty} \frac{H_k}{(4k+1)^3} + 2\sum_{k=1}^{\infty} \frac{H_k}{(4k+3)^3} \\
&= -\frac{1}{4} S_3\left(\frac{1}{2}\right) + \frac{1}{32}\left(S_3\left(\frac{1}{4}\right) + S_3\left(\frac{3}{4}\right)\right) \\
&= -\frac{1}{4}\left(-\frac{\pi^4}{8} - 14\ln(2)\zeta(3) + \frac{\pi^4}{4}\right) \\
&\quad + \frac{1}{32}\left(-\frac{(8G+\pi^2)^2}{2} + \frac{(3\ln(2)+\frac{\pi}{2})(-56\zeta(3)-2\pi^3)}{2} - \frac{(-8G+\pi^2)^2}{2} + \frac{(3\ln(2)-\frac{\pi}{2})(-56\zeta(3)+2\pi^3)}{2} + 4\pi^4\right) \\
&= -2G^2 + \frac{\pi^4}{32} - \frac{7\ln(2)\zeta(3)}{4}.
\end{aligned}$$

□