Problem 12227
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Proposed by G. Galperin and Y. J. Ionin (USA).

Prove that for any integer n with $n \geq 3$ there exist infinitely many pairs (A, B) such that A is a set of n consecutive positive integers, B is a set of fewer than n positive integers, A and B are disjoint, and

$$\sum_{k\in A} \frac{1}{k} = \sum_{k\in B} \frac{1}{k}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We distinguish four different cases.

$\boxed{\mathbf{n = 3m \text{ with } m \geq 1}}$ For all $N \geq 2$, by letting

$$A_N = \{k : 3N-1 \leq k \leq 3(N+m)-2\},$$
$$B_N = \{k : N \leq k \leq N+m-1\} \cup \left\{\frac{3k(3k-1)(3k+1)}{2} : N \leq k \leq N+m-1\right\},$$

we have that $n = 3m = |A_N| > 2m = |B_n|$, $A_N \cap B_N = \emptyset$, and $\sum_{k\in A_N} \frac{1}{k} = \sum_{k\in B_N} \frac{1}{k}$.
Note that $\frac{3k(3k-1)(3k+1)}{2}$ is integer because any triple of consecutive integers, at least one is even.

$\boxed{\mathbf{n = 4}}$ For all $N \geq 2$, by letting

$$A_N = \{4N-2, 4N-1, 4N, 4N+1\}, \quad B_N = \{N, 2N(4N-2), 2N(4N-1)(4N+1)\},$$

we have that $n = 4 = |A_N| > 3 = |B_n|$, $A_N \cap B_N = \emptyset$, and $\sum_{k\in A_N} \frac{1}{k} = \sum_{k\in B_N} \frac{1}{k}$.

$\boxed{\mathbf{n = 3m+1 \text{ with } m \geq 2}}$ For all $N \geq 2$, by letting

$$A_N = \{k : 3N-1 \leq k \leq 3(N+m)-1\},$$
$$B_N = \{k : N \leq k \leq N+m-1\} \cup \left\{\frac{3k(3k-1)(3k+1)}{2} : N \leq k \leq N+m-1\right\}$$
$$\cup \{3(N+m), 3(N+m)(3(N+m)-1)\},$$

we have that $n = 3m+1 = |A_N| > 2m+2 = |B_n|$, $A_N \cap B_N = \emptyset$, and $\sum_{k\in A_N} \frac{1}{k} = \sum_{k\in B_N} \frac{1}{k}$.

$\boxed{\mathbf{n = 3m+2 \text{ with } m \geq 1}}$ For all $N \geq 2$ such that $N+m$ is even, by letting

$$A_N = \{k : 3N-1 \leq k \leq 3(N+m)\},$$
$$B_N = \{k : N \leq k \leq N+m-1\} \cup \left\{\frac{3k(3k-1)(3k+1)}{2} : N \leq k \leq N+m-1\right\}$$
$$\cup \left\{\frac{3(N+m)}{2}, 3(N+m)(3(N+m)-1)\right\},$$

we have that $n = 3m+2 = |A_N| > 2m+2 = |B_n|$, $A_N \cap B_N = \emptyset$, and $\sum_{k\in A_N} \frac{1}{k} = \sum_{k\in B_N} \frac{1}{k}$.

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