Problem 12225
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Let Γ *denote the gamma function.*
(a) Prove that $\lceil\Gamma(1/n)\rceil = n$ *for every positive integer* n*.*
(b) Find the smallest constant c *such that* $\Gamma(1/n) \geq n - c$ *for every positive integer* n*.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. For $x > 0$, from $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt$, we find that

$$\Gamma''(x) = \int_0^\infty e^{-t}t^{x-1}(\ln(t))^2\,dt > 0,$$

which implies that the gamma function is strictly convex in $(0, +\infty)$.
Since $\Gamma(1) = \Gamma(2) = 1$ and $\Gamma'(1) = -\gamma$, by the strict convexity of Γ, it follows that

$$\forall x \in (1, 2], \qquad -\gamma(x-1) + 1 < \Gamma(x) \leq 1 \tag{1}$$

where $y = \Gamma'(1)(x-1) + \Gamma(1) = -\gamma(x-1) + 1$ is the tangent line to Γ at $x = 1$.

(a) Given any positive integer n, we have that $\lceil\Gamma(1/n)\rceil = n$ if and only if

$$n - 1 < \Gamma(1/n) \leq n \quad \Leftrightarrow \quad 1 - \frac{1}{n} < \frac{1}{n}\Gamma\left(\frac{1}{n}\right) = \Gamma\left(1 + \frac{1}{n}\right) \leq 1$$

which holds because by letting $x = 1 + \frac{1}{n} \in (1, 2]$ in (1), we get

$$1 - \frac{1}{n} < -\frac{\gamma}{n} + 1 < \Gamma\left(1 + \frac{1}{n}\right) \leq 1.$$

(b) By the Mean Value Theorem, for any positive integer n,

$$\Gamma(1/n) \geq n - c \quad \Leftrightarrow \quad c \geq -\frac{\Gamma\left(1 + \frac{1}{n}\right) - \Gamma(1)}{\left(1 + \frac{1}{n}\right) - 1} = -\Gamma'(x_n) \quad \text{for some } x_n \in \left(1, 1 + \frac{1}{n}\right).$$

Therefore c is the smallest constant such that $\Gamma(1/n) \geq n - c$ for all $n \in \mathbb{N}^+$, if and only if

$$c = \sup_{n\in\mathbb{N}^+}(-\Gamma'(x_n)) = -\inf_{n\in\mathbb{N}^+}\Gamma'(x_n) = -\lim_{x\to 1^+}\Gamma'(x) = -\Gamma'(1) = \gamma$$

where we applied the fact that, by the strict convexity of Γ, Γ' is a strictly increasing continuous function. $\square$