Problem 12224
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Proposed by C. Perng (USA).

Let ABC be a triangle, with D and E on AB and AC, respectively. For a point F in the plane, let DF intersect BC at G and let EF intersect BC at H. Furthermore, let AF intersect BC at I, let DH intersect EG at J, and let BE intersect CD at K. Prove that I, J, and K are collinear.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We consider the triangle ABC in the complex plane. Without loss of generality we may assume that $B = -1$, $C = 1$, $A = x + iy$ and $F = u + iv$ with $y \neq 0$ and $v \neq y$.
Moreover let

$$D = tA + (1-t)B = -1 + t + t(x+iy) \quad \text{and} \quad E = sA + (1-s)C = 1 - s + s(x+iy)$$

with $t, s \in (0,1)$, $v \neq ty$, and $v \neq sy$.
It is known that when X, Y, U, V are points in the complex plane then the lines XY and UV, if they are not parallel, intersect at

$$\frac{(\overline{X}Y - X\overline{Y})(U-V) - (X-Y)(\overline{U}V - U\overline{V})}{(\overline{X}-\overline{Y})(U-V) - (X-Y)(\overline{U}-\overline{V})}.$$

By applying the above formula we find

$$G = \frac{v - txv + tyu - tv}{ty - v} \quad , \quad H = \frac{-v - sxv + syu + sv}{sy - v} \quad , \quad I = \frac{yu - xv}{y - v}$$
$$J = \frac{st(xv - yu) + v(t - s + st(x+iy))}{(t+s)v - sty} \quad , \quad K = \frac{t - s + st(x+iy)}{t + s - st} .$$

Then, it is easy to verify that $I = rJ + (1-r)K$ with

$$r = \frac{sty - (s+t)v}{st(y-v)} \in \mathbb{R}$$

which implies that I, J, and K are collinear. $\square$