**Problem 12222**
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*Prove*

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\sum_{n=k}^{\infty}\frac{1}{n2^n}=-\frac{13\zeta(3)}{24}.$$

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*Solution.* Since

$$\int_0^1 t^k\log(t)\,dt=\left[\frac{t^{k+1}}{k+1}\log(t)\right]_0^1-\int_0^1\frac{t^{k+1-1}}{k+1}\,dt=-\frac{1}{(k+1)^2},$$

it follows that the given double series has an integral representation

$$\begin{aligned}
I:=\int_0^1\frac{\log(t+2)\,\log(t)}{t+1}\,dt&=\int_0^1\frac{(\log(2)+\log(1+\frac{t}{2}))\,\log(t)}{t+1}\,dt\\
&=\log(2)\int_0^1\frac{\log(t)}{t+1}\,dt+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n2^n}\int_0^1\frac{t^n\log(t)}{t+1}\,dt\\
&=\log(2)\int_0^1\log(t)\sum_{k=0}^{\infty}(-1)^kt^k\,dt+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n2^n}\int_0^1 t^n\log(t)\sum_{k=0}^{\infty}(-1)^kt^k\,dt\\
&=-\log(2)\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n2^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{(n+k+1)^2}\\
&=-\frac{\log(2)\zeta(2)}{2}+\sum_{n=1}^{\infty}\frac{1}{n2^n}\left(\frac{\zeta(2)}{2}+\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\right)\\
&=-\frac{\log(2)\zeta(2)}{2}+\frac{\zeta(2)}{2}\sum_{n=1}^{\infty}\frac{1}{n2^n}+\sum_{n=1}^{\infty}\frac{1}{n2^n}\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\\
&=-\frac{\log(2)\zeta(2)}{2}+\frac{\log(2)\zeta(2)}{2}+\sum_{n=1}^{\infty}\frac{1}{n2^n}\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\\
&=\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\sum_{n=k}^{\infty}\frac{1}{n2^n}.
\end{aligned}$$

Now we evaluate the integral $I$. We introduce two ausiliary functions $F_-$ and $F_+$:

$$\begin{aligned}
F_-(x):=\int_0^x\frac{\log^2(t)}{1-t}\,dt&=-\log^2(x)\log(1-x)+2\int_0^x\frac{\log(t)\log(1-t)}{t}\,dt\\
&=-\log^2(x)\log(1-x)-2\int_0^x\log(t)\sum_{k=1}^{\infty}\frac{t^{k-1}}{k}\,dt\\
&=-\log^2(x)\log(1-x)-2\log(x)\mathrm{Li}_2(x)+2\int_0^x\sum_{k=1}^{\infty}\frac{t^{k-1}}{k^2}\,dt\\
&=-\log^2(x)\log(1-x)-2\log(x)\mathrm{Li}_2(x)+2\mathrm{Li}_3(x)
\end{aligned}$$

and, in a similar way,

$$F_+(x):=\int_0^x\frac{\log^2(t)}{1+t}\,dt=\log^2(x)\log(1+x)+2\log(x)\mathrm{Li}_2(-x)-2\mathrm{Li}_3(-x).$$

Hence

$$I=\int_0^1\frac{\log(t+2)\,\log(t)}{t+1}\,dt=\frac{1}{2}\int_0^1\left(\frac{\log^2(t)}{t+1}+\frac{\log^2(t+2)}{t+1}-\frac{\log^2\left(\frac{t}{t+2}\right)}{t+1}\right)\,dt.$$

The first term is

$$\int_0^1 \frac{\log^2(t)}{t+1}\,dt = F_+(1) = -2\mathrm{Li}_3\left(-1\right) = \frac{3\zeta(3)}{2}.$$

Moreover, by letting $s = 1/(t+2)$ we get

$$\begin{aligned}\int_0^1 \frac{\log^2(t+2)}{t+1}\,dt &= -\int_{1/2}^{1/3} \frac{\log^2(s)}{s(1-s)}\,ds = \int_{1/3}^{1/2} \frac{\log^2(s)}{1-s}\,ds + \int_{1/3}^{1/2} \frac{\log^2(s)}{s}\,ds \\ &= F_-(1/2) - F_-(1/3) + \frac{\log^3(3)}{3} - \frac{\log^3(2)}{3}\end{aligned}$$

and, by letting $s = t/(t+2)$ we find

$$\begin{aligned}\int_0^1 \frac{\log^2\left(\frac{t}{t+2}\right)}{t+1}\,dt &= 2\int_0^{1/3} \frac{\log^2(s)}{1-s^2}\,ds = \int_0^{1/3} \frac{\log^2(s)}{1-s}\,ds + \int_0^{1/3} \frac{\log^2(s)}{1+s}\,ds \\ &= F_-(1/3) + F_+(1/3).\end{aligned}$$

Therefore

$$\begin{aligned}I &= \frac{3\zeta(3)}{4} + \frac{\log^3(3)}{6} - \frac{\log^3(2)}{6} + \frac{F_-(1/2)}{2} - F_-(1/3) - \frac{F_+(1/3)}{2} \\ &= \frac{13\zeta(3)}{8} - \frac{\log^3(3)}{3} - \log(3)\left(2\mathrm{Li}_2\left(\frac{1}{3}\right) - \mathrm{Li}_2\left(-\frac{1}{3}\right)\right) - \left(2\mathrm{Li}_3\left(\frac{1}{3}\right) - \mathrm{Li}_3\left(-\frac{1}{3}\right)\right).\end{aligned}$$

Since it is known that (see Lewin's book *Polylogarithms and Associated Functions*)

$$2\mathrm{Li}_2\left(\frac{1}{3}\right) - \mathrm{Li}_2\left(-\frac{1}{3}\right) = \frac{\pi^2}{6} - \frac{\log^2(3)}{2}$$

and

$$2\mathrm{Li}_3\left(\frac{1}{3}\right) - \mathrm{Li}_3\left(-\frac{1}{3}\right) = \frac{\log^3(3)}{6} - \frac{\log(3)\pi^2}{6} + \frac{13\zeta(3)}{6}$$

we finally find

$$I = -\frac{13\zeta(3)}{24}.$$

□