Problem 12221
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Proposed by N. Batir (Turkey).

Prove

$$\int_0^1 \frac{\log(x^6+1)}{x^2+1}\,dx = \frac{\pi\log(6)}{2} - 3G,$$

where G is the Catalan's constant $\sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. By letting $x = 1/t$,

$$I := \int_0^1 \frac{\log(x^6+1)}{x^2+1}\,dx = \int_1^{+\infty} \frac{\log(1/t^6+1)}{t^2+1}\,dt = \int_1^{+\infty} \frac{\log(t^6+1)}{t^2+1}\,dt - 6\int_1^{+\infty}\frac{\log(t)}{t^2+1}\,dt.$$

We have

$$\int_1^{\infty} \frac{\log(t)}{t^2+1}\,dt = -\int_0^1 \frac{\log(x)}{1+x^2}\,dx = -\int_0^1 \log(x)\,d(\arctan(x)) = [-\log(x)\arctan(x)]_0^1 + \int_0^1 \frac{\arctan(x)}{x}\,dx$$
$$= 0 + \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2k+1}\,dx = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} = G.$$

Hence, since $x^6+1 = (x^2+1)(x^2+\sqrt{3}x+1)(x^2-\sqrt{3}x+1)$, we find

$$I = \frac{1}{2}\int_0^1 \frac{\log(x^6+1)}{x^2+1}\,dx + \frac{1}{2}\int_1^{+\infty}\frac{\log(t^6+1)}{t^2+1}\,dt - 3\int_1^{+\infty}\frac{\log(t)}{t^2+1}\,dt = \frac{1}{2}\int_0^{+\infty}\frac{\log(x^6+1)}{x^2+1}\,dx - 3G$$
$$= \frac{J(0)+J(\sqrt{3})+J(-\sqrt{3})}{2} - 3G,$$

where, for $|a|<2$,

$$J(a) = \int_0^{+\infty} \frac{\log(x^2+ax+1)}{x^2+1}\,dx = \int_0^{\pi/2} \log\left(\frac{1}{\cos^2(t)} + a\tan(t)\right)dt$$
$$= \int_0^{\pi/2}\log\left(1+\frac{a}{2}\sin(2t)\right)dt - 2\int_0^{\pi/2}\log(\cos(t))\,dt$$
$$= \int_0^{\pi/2}\log\left(1+\frac{a}{2}\sin(t)\right)dt + \pi\log(2).$$

In the last step we applied $\int_0^{\pi/2}\log(\cos(t))\,dt = -\frac{\pi}{2}\log(2)$ which follows from

$$\int_0^{\pi/2}\log(\cos(t))\,dt = \int_0^{\pi/2}\log(\sin(t))\,dt = \frac{1}{2}\int_0^{\pi/2}\log(\sin(2t)/2)\,dt = \frac{1}{2}\int_0^{\pi/2}\log(\sin(t))\,dt - \frac{\pi}{4}\log(2).$$

By using the Wallis' formula $\int_0^{\frac{\pi}{2}}\sin^{2n}(x)\,dx = \frac{\pi}{2}\frac{1}{4^n}\binom{2n}{n}$,

$$J(a)+J(-a) = \int_0^{\pi/2}\log\left(1-\frac{a^2}{4}\sin^2(t)\right)dt + 2\pi\log(2)$$
$$= -\sum_{n=1}^{\infty}\frac{(a^2/4)^n}{n}\int_0^{\pi/2}\sin^{2n}(t)\,dt + 2\pi\log(2)$$
$$= -\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{a^{2n}}{16^n n}\binom{2n}{n} + 2\pi\log(2) = \pi\log\left(\frac{1+\sqrt{1-a^2/4}}{2}\right) + 2\pi\log(2).$$

Finally,

$$I = \frac{J(0)+J(\sqrt{3})+J(-\sqrt{3})}{2} - 3G = \frac{\pi\log(2)+\pi\log\left(\frac{3}{4}\right)+2\pi\log(2)}{2} - 3G = \frac{\pi\log(6)}{2} - 3G.$$

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