Problem 12220
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Proposed by D. M. Batinetu-Giurgiu and N. Stanciu (Romania).

Let $a_n = \sum_{k=1}^n 1/k^2$ *and* $b_n = \sum_{k=1}^n 1/(2k-1)^2$. *Prove*

$$\lim_{n\to\infty} n\left(\frac{b_n}{a_n} - \frac{3}{4}\right) = \frac{3}{\pi^2}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Since $b_n = a_{2n} - \frac{a_n}{4}$, as $n \to \infty$,

$$n\left(\frac{b_n}{a_n} - \frac{3}{4}\right) = n\left(\frac{a_{2n}}{a_n} - 1\right) = \frac{n}{a_n}\sum_{k=1}^n \frac{1}{(n+k)^2} = \frac{1}{a_n}\cdot\frac{1}{n}\sum_{k=1}^n \frac{1}{(1+k/n)^2} \to \frac{3}{\pi^2}$$

because

$$a_n \to \frac{\pi^2}{6} \quad \text{and} \quad \frac{1}{n}\sum_{k=1}^n \frac{1}{(1+k/n)^2} \to \int_0^1 \frac{dx}{(1+x)^2} = \frac{1}{2}.$$

□