Problem 12219
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Let k and m be positive integers with $k < m$. Let $c(m,k)$ be the number of permutations of $\{1,\ldots,m\}$ consisting of k cycles. The numbers $c(m,k)$ are known as unsigned Stirling numbers of the first kind. Prove

$$\sum_{j=k}^{m} \frac{(-2)^j \binom{m}{j} c(j,k)}{(j-1)!} = 0$$

whenever m and k have opposite parity.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let

$$F_m(x) = \sum_{k=1}^{m} (-x)^k \sum_{j=k}^{m} \frac{(-2)^j \binom{m}{j} c(j,k)}{(j-1)!}$$

then it suffices to show that the polynomial F_m is even when m is even and it is odd when m is odd, that is

$$F_m(-x) = (-1)^m F_m(x).$$

We have that

$$\begin{aligned}
F_m(x) &:= \sum_{j=1}^{m} \frac{2^j \binom{m}{j}}{(j-1)!} \sum_{k=1}^{j} (-1)^{j-k} c(j,k) x^k \\
&= \sum_{j=1}^{m} \frac{2^j \binom{m}{j}}{(j-1)!} \cdot x(x-1)\cdots(x-j+1) \\
&= m \sum_{j=1}^{m} 2^j \binom{m-1}{j-1} \binom{x}{j} = m \sum_{j=0}^{m} 2^j \binom{m-1}{m-j} \binom{x}{j} \\
&= m[z^m]\,(1+z)^{m-1}(1+2z)^x \\
&= m[z^m]\,(1+z)^{x+m-1} \left(1 + \frac{z}{1+z}\right)^x \\
&= m \sum_{j=0}^{m} \binom{x+m-j-1}{m-j} \binom{x}{j}.
\end{aligned}$$

Therefore, since $\binom{-y}{r} = (-1)^r \binom{y+r-1}{r}$, it follows that

$$\begin{aligned}
F_m(-x) &:= m \sum_{j=0}^{m} \binom{-x+m-j-1}{m-j} \binom{-x}{j} = m \sum_{j=0}^{m} \binom{-(x-j+1)}{j} \binom{-x}{m-j} \\
&= m \sum_{j=0}^{m} (-1)^j \binom{x}{j} \cdot (-1)^{m-j} \binom{x+m-j-1}{j} = (-1)^m F_m(x)
\end{aligned}$$

and the proof is complete. □