Problem 12218
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For which positive integers n does there exist an ordering of all permutations of $\{1, \ldots, n\}$ so that their composition in that order is the identity?

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Solution. We claim that such ordering exists for $n = 1$ and for all positive integers $n \geq 4$.

For $n = 1$ the ordering exists, trivially. For $n > 1$, there are $n!/2$ permutations with signature 1 and $n!/2$ permutations with signature -1. It follows that the composition of all permutations of S_n has signature $(-1)^{n!/2}$, which is -1 for $n = 2$ and for $n = 3$ and therefore the it cannot be the identity, whose signature is 1.

Now we assume that $n \geq 4$. We may always arrange the terms in such a way that each permutation of order > 2 has its inverse (which is a different permutation of the same order) next to it. Hence it suffices to find an ordering for $I_n(1, 2, \ldots, n)$, the set of involutions of S_n, that is, the set of permutations of order ≤ 2 (where we include among involutions the identity permutation). More specifically, here $I_n(a_1, a_2, \ldots, a_n)$ means the set of involutions on the symbols $a_1, a_2, \ldots, a_n$. Note that $|I_n| = |I_{n-1}| + (n-1)|I_{n-2}|$ and since $|I_2| = 2$ and $|I_3| = 4$, we deduce that $|I_n|$ is even for all $n \geq 2$.

For $n = 4$, a *good* ordering in $I_4(1, 2, 3, 4)$ is (from left to right):

$$() \ , \ (1,2) \ , \ (3,4) \ , \ (1,2)(3,4) \ , \ (1,4) \ , \ (2,3) \ , \ (1,4)(2,3) \ , \ (1,3) \ , \ (2,4) \ , \ (1,3)(2,4) \ .$$

For $n = 5$, a *good* ordering in $I_5(1, 2, 3, 4, 5)$ is:

$$\begin{aligned}
&() \ , \ (1,2) \ , \ (3,4) \ , \ (1,2)(3,4) \ , \ (1,4) \ , \ (2,3) \ , \ (1,4)(2,3) \ , \ (1,3) \ , \ (2,4) \ , \ (1,3)(2,4) \ , \\
&(1,5) \ , \ (1,5)(2,3) \ , \ (1,5)(2,4) \ , \ (1,5)(3,4) \ , \ (2,5) \ , \ (1,3)(2,5) \ , \ (2,5)(3,4) \ , \ (1,4)(2,5) \ , \\
&(3,5) \ , \ (1,2)(3,5) \ , \ (2,4)(3,5) \ , \ (1,4)(3,5) \ , \ (4,5) \ , \ (1,2)(4,5) \ , \ (2,3)(4,5) \ , \ (1,3)(4,5) \ .
\end{aligned}$$

For $n > 5$, a *good* ordering in $I_n(1, 2, \ldots, n)$ can be recursively defined based on the orderings of I_{n-1} and I_{n-2} in the following way,

$$I_n(1, 2, \ldots, n) = (n) I_{n-1}(1, 2, \ldots, n-1) + \sum_{k=1}^{n-1} (k, n) I_{n-2}(1, 2, \ldots, \widehat{k}, \ldots, n-1)$$

where $\widehat{k}$ means that k is omitted. Here, by sum of orderings on disjoint sets we mean juxtaposition.

Thus, starting with the ordered sequence $1, 2, \ldots, n$, the first $|I_{n-1}|$ permutations of the induced ordering on I_n keep n fixed, while the other elements are permuted under the action of I_{n-1}, and the final result is the ordered sequence $1, 2, \ldots, n$. In other words, the product of the first $|I_{n-1}|$ permutations equals the identity permutation.

Then follow $n-1$ sets of involutions. In the k-th sets, the transposition (k, n) swaps the elements k and n a number $|I_{n-2}|$ of times which is an even number, while the other elements are permuted under the action of I_{n-2}, obtaining again the ordered sequence $1, 2, \ldots, n$. This shows that the induced ordering on I_n satisfies the desired property and the proof is complete. $\square$