Problem 12217
(American Mathematical Monthly, Vol.127, December 2020)

Proposed by G. Fera (Italy).

Let I be the incenter and G be the centroid of a triangle ABC. Prove

$$\frac{3}{2} < \frac{|AI|}{|AG|} + \frac{|BI|}{|BG|} + \frac{|CI|}{|CG|} \leq 3.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let m_a, m_b, m_c be, respectively, the lengths of the medians of the triangle ABC. It is easy to verify that

$$|AI|^2 = \frac{bc(s-a)}{s}, \quad |AG| = \frac{2m_a}{3}, \quad m_a^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4} = s(s-a) + \frac{(b-c)^2}{4}.$$

Similar formulas hold for $|BI|$, m_b, and $|CI|$, m_c.

Upper bound. Since

$$m_a^2 \geq s(s-a), \quad m_b^2 \geq s(s-b), \quad m_c^2 \geq s(s-c),$$

it follows that

$$\begin{aligned}
\frac{|AI|}{|AG|} + \frac{|BI|}{|BG|} + \frac{|CI|}{|CG|} &\leq \sqrt{3}\left(\frac{|AI|^2}{|AG|^2} + \frac{|BI|^2}{|BG|^2} + \frac{|CI|^2}{|CG|^2}\right)^{1/2} \\
&= 3\left(\frac{3bc(s-a)}{4sm_a^2} + \frac{3ca(s-b)}{4sm_b^2} + \frac{3ab(s-c)}{4sm_c^2}\right)^{1/2} \\
&\leq 3\left(\frac{3(ab+bc+ca)}{(a+b+c)^2}\right)^{1/2} \leq 3
\end{aligned}$$

where at the last step we used $3(ab+bc+ca) \leq (a+b+c)^2$.

Lower bound. We have that

$$|AI| = \sqrt{(s-a)^2 + r^2} > s-a$$

where r is the inradius, and by the triangle inequality

$$\begin{cases} m_a \leq \frac{a}{2} + b \\ m_a \leq \frac{a}{2} + c \end{cases} \implies m_a \leq \frac{a+b+c}{2} = s \implies |AG| \leq \frac{2s}{3}.$$

Similar inequalities hold for $|BI|$, $|BG|$ and for $|CI|$, $|CG|$. Hence

$$\frac{|AI|}{|AG|} + \frac{|BI|}{|BG|} + \frac{|CI|}{|CG|} > \frac{3(s-a)}{2s} + \frac{3(s-b)}{2s} + \frac{3(s-c)}{2s} = \frac{3}{2}.$$

$\square$