Problem 12215
(American Mathematical Monthly, Vol.127, November 2020)

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Calculate

$$\sum_{n=1}^{\infty}\left(\left(\frac{1}{n^2}+\frac{1}{(n+2)^2}+\frac{1}{(n+4)^2}+\cdots\right)-\frac{1}{2n}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that

$$\begin{aligned}S&=\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}\frac{1}{(n+2k)^2}-\frac{1}{2n}\right)=\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}\frac{1}{(2n+2k)^2}-\frac{1}{4n}\right)+\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}\frac{1}{(2n-1+2k)^2}-\frac{1}{4n-2}\right)\\&=\frac{1}{4}\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{1}{k^2}-\frac{1}{n}\right)+\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{1}{(2k-1)^2}-\frac{1}{4n-2}\right)=\frac{1}{4}+\left(\frac{1}{4}+\frac{\pi^2}{16}\right)=\frac{1}{2}+\frac{\pi^2}{16}\end{aligned}$$

because, as $N\to\infty$,

$$\begin{aligned}\sum_{n=1}^{N}\left(\sum_{k=n}^{\infty}\frac{1}{k^2}-\frac{1}{n}\right)&=\sum_{n=1}^{N}\left(\left(\frac{\pi^2}{6}-\sum_{k=1}^{n-1}\frac{1}{k^2}\right)-\frac{1}{n}\right)\\&=N\,\frac{\pi^2}{6}-\sum_{n=1}^{N}\sum_{k=1}^{n-1}\frac{1}{k^2}-\sum_{n=1}^{N}\frac{1}{n}\\&=N\,\frac{\pi^2}{6}-\sum_{k=1}^{N-1}\frac{1}{k^2}\sum_{n=k+1}^{N}1-\sum_{n=1}^{N}\frac{1}{n}\\&=N\,\frac{\pi^2}{6}-\sum_{k=1}^{N-1}\frac{N-k}{k^2}-\sum_{n=1}^{N}\frac{1}{n}\\&=N\left(\frac{\pi^2}{6}-\sum_{k=1}^{N-1}\frac{1}{k^2}\right)-\frac{1}{N}\to 1,\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{N}\left(\sum_{k=n}^{\infty}\frac{1}{(2k-1)^2}-\frac{1}{4n-2}\right)&=\sum_{n=1}^{N}\left(\left(\frac{\pi^2}{8}-\sum_{k=1}^{n-1}\frac{1}{(2k-1)^2}\right)-\frac{1}{4n-2}\right)\\&=N\,\frac{\pi^2}{8}-\sum_{n=1}^{N}\sum_{k=1}^{n-1}\frac{1}{(2k-1)^2}-\frac{1}{2}\sum_{n=1}^{N}\frac{1}{2n-1}\\&=N\,\frac{\pi^2}{8}-\sum_{k=1}^{N-1}\frac{1}{(2k-1)^2}\sum_{n=k+1}^{N}1-\frac{1}{2}\sum_{n=1}^{N}\frac{1}{2n-1}\\&=N\,\frac{\pi^2}{8}-\sum_{k=1}^{N-1}\frac{N-k}{(2k-1)^2}-\frac{1}{2}\sum_{n=1}^{N}\frac{1}{2n-1}\\&=N\left(\frac{\pi^2}{8}-\sum_{k=1}^{N-1}\frac{1}{(2k-1)^2}\right)+\frac{1}{2}\sum_{k=1}^{N-1}\frac{1}{(2k-1)^2}-\frac{1/2}{2N-1}\to\frac{1}{4}+\frac{\pi^2}{16}.\end{aligned}$$

Note that by Stolz-Cesaro Theorem,

$$\lim_{N\to\infty}N\left(\frac{\pi^2}{6}-\sum_{k=1}^{N-1}\frac{1}{k^2}\right)=\lim_{N\to\infty}\frac{-\frac{1}{N^2}}{\frac{1}{N+1}-\frac{1}{N}}=1,$$

$$\lim_{N\to\infty}N\left(\frac{\pi^2}{8}-\sum_{k=1}^{N-1}\frac{1}{(2k-1)^2}\right)=\lim_{N\to\infty}\frac{-\frac{1}{(2N-1)^2}}{\frac{1}{N+1}-\frac{1}{N}}=\frac{1}{4}.$$

□