Problem 12213
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Proposed by H. Ohtsuka (Japan).

For $n \geq 1$, prove

$$\sum_{k=1}^{n} \sqrt{F_{k-1}F_{k+2}} \leq \sqrt{F_{n+1}F_{n+4}} - \sqrt{5}$$

where F_n is the n-th Fibonacci number.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. For $n = 1$ the equality holds. For $n \geq 2$, we show that the strict inequality holds

$$\sum_{k=1}^{n} \sqrt{F_{k-1}F_{k+2}} < \sqrt{F_{n+1}F_{n+4}} - \sqrt{5}.$$

It suffices to show that for $k \geq 2$,

$$\sqrt{F_{k-1}F_{k+2}} < \sqrt{F_{k+1}F_{k+4}} - \sqrt{F_kF_{k+3}} \tag{1}$$

then

$$\sum_{k=1}^{n} \sqrt{F_{k-1}F_{k+2}} = \sum_{k=2}^{n} \sqrt{F_{k-1}F_{k+2}} < \sum_{k=2}^{n} \left(\sqrt{F_{k+1}F_{k+4}} - \sqrt{F_kF_{k+3}}\right) = \sqrt{F_{n+1}F_{n+4}} - \sqrt{F_2F_5}$$

and the given inequality follows because $F_0F_3 = 0$ and $F_2F_5 = 5$.
The inequality (1) is equivalent to

$$F_{k-1}F_{k+2} < F_{k+1}F_{k+4} + F_kF_{k+3} - 2\sqrt{F_kF_{k+1}F_{k+3}F_{k+4}}$$

that is

$$\begin{aligned} 2\sqrt{F_kF_{k+1}F_{k+3}F_{k+4}} &< F_{k+1}F_{k+4} + F_kF_{k+3} - F_{k-1}F_{k+2} \\ &= (F_{k+2} - F_k)(F_{k+2} + F_{k+3}) + F_kF_{k+3} - F_{k-1}F_{k+2} \\ &= F_{k+2}(F_{k+2} + F_{k+3} - F_k - F_{k-1}) = 2F_{k+2}^2. \end{aligned}$$

Hence it remains to show that

$$F_kF_{k+1}F_{k+3}F_{k+4} < F_{k+2}^4$$

which holds, because by Cassini's identity $F_{n-r}F_{n+r} = F_n^2 - (-1)^{n-r}F_r^2$,

$$\begin{aligned} F_kF_{k+1}F_{k+3}F_{k+4} &= (F_kF_{k+4})(F_{k+1}F_{k+3}) \\ &= (F_{k+2}^2 - (-1)^k)(F_{k+2}^2 + (-1)^k) \\ &= F_{k+2}^4 - 1 < F_{k+2}^4. \end{aligned}$$

□