Problem 12210
(American Mathematical Monthly, Vol.127, November 2020)

Proposed by P. Bracken (USA).

Let $x_1 = 1$, *and let*

$$x_{n+1} = \left(\sqrt{x_n} + \frac{1}{\sqrt{x_n}}\right)^2$$

when $n \geq 1$. *For* $n \geq 1$, *let*

$$a_n = 2n + \frac{\log(n)}{2} - x_n.$$

Show that the sequence $(a_n)_{n\geq 1}$ *converges.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. For $n \geq 1$,

$$x_{n+1} = \left(\sqrt{x_n} + \frac{1}{\sqrt{x_n}}\right)^2 = x_n + \frac{1}{x_n} + 2.$$

Let $y_n = x_n - 2n$ then $y_1 = -1$, $y_2 = 0$, and $y_{n+1} - y_n = \frac{1}{y_n + 2n}$. Therefore, for $n \geq 3$,

$$y_n = \sum_{k=2}^{n-1} \frac{1}{y_k + 2k} \implies 0 < y_n \leq \sum_{k=2}^{n-1} \frac{1}{2k} = \frac{H_{n-1} - 1}{2}.$$

Hence, for $n \geq 3$,

$$\begin{aligned} a_n &= \frac{\log(n)}{2} - y_n = \frac{\log(n)}{2} - \sum_{k=2}^{n-1} \frac{1}{y_k + 2k} \\ &= \frac{1}{2}\left(\log(n) - H_{n-1} + 1\right) + \frac{1}{2}\sum_{k=2}^{n-1} \frac{1}{k}\left(1 - \frac{1}{1 + \frac{y_k}{2k}}\right) \\ &\overset{n\to\infty}{\longrightarrow} \frac{1-\gamma}{2} + \frac{1}{2}\sum_{k=2}^{\infty} \frac{1}{k}\left(1 - \frac{1}{1 + \frac{y_k}{2k}}\right) \end{aligned}$$

because

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \log(n) + \gamma + o(1),$$

and, for $t \geq 0$, $0 \leq 1 - \frac{1}{1+t} \leq t$ implies

$$\sum_{k=2}^{\infty} \frac{1}{k} \underbrace{\left(1 - \frac{1}{1 + \frac{y_k}{2k}}\right)}_{\geq 0} \leq \sum_{k=2}^{\infty} \frac{1}{k} \cdot \frac{y_k}{2k} \leq \frac{1}{4}\sum_{k=2}^{\infty} \frac{H_{k-1} - 1}{k^2}$$

which is convergent since $\frac{H_{k-1}-1}{k^2} \sim \frac{\log(k)}{k^2}$. $\square$