Problem 12207
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Proposed by O. Furdui and A. Sintamarian (Romania).

Let $f : [0, 1] \to \mathbb{R}$ *be a continuous function satisfying* $\int_0^1 f(x)\,dx = 1$. *Evaluate*

$$\lim_{n\to\infty} \frac{n}{\ln(n)} \int_0^1 x^n f(x^n) \ln(1-x)\,dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $t = x^n$, then $x = t^{1/n}$, $dx = t^{1/n-1}dt/n$ and

$$\frac{n}{\ln(n)} \int_0^1 x^n f(x^n) \ln(1-x)\,dx = -\int_0^1 f(t)u_n(t)\,dt$$

where

$$u_n(t) = -\frac{t^{1/n}\ln(1-t^{1/n})}{\ln(n)}.$$

For all $t \in (0, 1)$,

$$\lim_{n\to\infty} u_n(t) = -\lim_{n\to\infty} \frac{\ln(1-e^{\ln(t)/n})}{\ln(n)} = -\lim_{n\to\infty} \frac{\ln(-\ln(t)/n)}{\ln(n)} = -\lim_{n\to\infty} \frac{\ln(-\ln(t)) - \ln(n)}{\ln(n)} = 1.$$

Moreover, for $n \geq 3$,

$$0 \leq u_n(t) \leq -\frac{\ln(1-t^{1/n})}{\ln(n)} \leq -\frac{\ln((1-t)/n)}{\ln(n)} \leq -\ln(1-t) + 1$$

and $\int_0^1 (-\ln(1-t) + 1)\,dt = 2$.
Therefore, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \frac{n}{\ln(n)} \int_0^1 x^n f(x^n) \ln(1-x)\,dx = -\lim_{n\to\infty} \int_0^1 f(t)u_n(t)\,dt = -\int_0^1 f(t)\,dt = -1.$$

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