Problem 12206
(American Mathematical Monthly, Vol.127, October 2020)

Proposed by S. Stewart (Australia).

Prove

$$\sum_{n=1}^{\infty}\frac{\overline{H}_{2n}}{n^2}=\frac{3}{4}\,\zeta(3)$$

where $\overline{H}_n$ *is the n-th skew-harmonic number* $\sum_{k=1}^{n}(-1)^{k-1}/k$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $H_n(x)=\sum_{k=1}^{n}x^k/k$. Then $H_n(1)=H_n$ and $H_n(-1)=-\overline{H}_n$.
Moreover, for $0<|x|\leq 1$,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n x^n}{n^2}&=\sum_{n=1}^{\infty}\frac{x^n}{n^2}\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{n+k}\right)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{x^n}{nk(n+k)}\\&=\sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty}\frac{x^{n-k}}{(n-k)kn}=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\frac{x^{n-k}}{(n-k)kn}=\sum_{n=2}^{\infty}\frac{1}{n^2}\sum_{k=1}^{n-1}\left(\frac{x^{n-k}}{k}+\frac{x^{n-k}}{n-k}\right)\\&=\sum_{n=1}^{\infty}\frac{H_{n-1}(1/x)x^n}{n^2}+\sum_{n=1}^{\infty}\frac{H_{n-1}(x)}{n^2}.\end{aligned}$$

Therefore, for $x=1$ we get

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2}=2\sum_{n=1}^{\infty}\frac{H_{n-1}}{n^2}=2\sum_{n=1}^{\infty}\frac{H_n}{n^2}-2\zeta(3)\implies\sum_{n=1}^{\infty}\frac{H_n}{n^2}=2\,\zeta(3),$$

and for $x=-1$ we find

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^2}&=-\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_{n-1}}{n^2}-\sum_{n=1}^{\infty}\frac{\overline{H}_{n-1}}{n^2}=-\sum_{n=1}^{\infty}\frac{((-1)^n+1)\overline{H}_{n-1}}{n^2}\\&=-2\sum_{n=1}^{\infty}\frac{\overline{H}_{2n-1}}{(2n)^2}=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\overline{H}_{2n}}{n^2}-\frac{1}{4}\,\zeta(3).\end{aligned}\tag{1}$$

On the other hand,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\overline{H}_{2n}}{n^2}&=\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sum_{k=1}^{n}\frac{1}{2k-1}-\sum_{k=1}^{n}\frac{1}{2k}\right)=\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\left(H_{2n}-\frac{H_n}{2}\right)-\frac{H_n}{2}\right)\\&=4\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n)^2}-\sum_{n=1}^{\infty}\frac{H_n}{n^2}=2\sum_{n=1}^{\infty}\frac{((-1)^n+1)H_n}{n^2}-\sum_{n=1}^{\infty}\frac{H_n}{n^2}\\&=2\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^2}+\sum_{n=1}^{\infty}\frac{H_n}{n^2}=2\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^2}+2\,\zeta(3).\end{aligned}\tag{2}$$

By solving the linear system given by (1) and (2) we easily obtain

$$\sum_{n=1}^{\infty}\frac{\overline{H}_{2n}}{n^2}=\frac{3}{4}\,\zeta(3),\quad\text{and}\quad\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^2}=-\frac{5}{8}\,\zeta(3).$$

Remark: all the above infinite series are absolutely convergent and therefore the terms can be rearranged in a different order. □