Problem 12205
(American Mathematical Monthly, Vol.127, October 2020)

Proposed by C. Chiser (Romania).

Find the minimum value of

$$\frac{\int_0^1 x^2(f'(x))^2\,dx}{\int_0^1 x^2(f(x))^2\,dx}$$

over all nonzero continuously differentiable functions $f:[0,1]\to\mathbb{R}$ *with* $f(1)=0$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $g(x)=xf(x)$ then $g\in C^1([0,1])$, it is non-identically zero and $g(0)=g(1)=0$. Moreover, by integrating by parts we get

$$\begin{aligned}\int_0^1 x^2(f'(x))^2\,dx &= \int_0^1 (g'(x)-f(x))^2\,dx\\ &= \int_0^1 (g'(x))^2\,dx - 2\int_0^1 f(x)\,d(g(x)) + \int_0^1 (f(x))^2\,dx\\ &= \int_0^1 (g'(x))^2\,dx - 2[f(x)g(x)]_0^1 + \int_0^1 x\,d(f(x))^2) + \int_0^1 (f(x))^2\,dx\\ &= \int_0^1 (g'(x))^2\,dx + [x(f(x))^2]_0^1 - \int_0^1 (f(x))^2\,dx + \int_0^1 (f(x))^2\,dx\\ &= \int_0^1 (g'(x))^2\,dx.\end{aligned}$$

Hence, by Wirtinger's inequality,

$$\frac{\int_0^1 x^2(f'(x))^2\,dx}{\int_0^1 x^2(f(x))^2\,dx} = \frac{\int_0^1 (g'(x))^2\,dx}{\int_0^1 (g(x))^2\,dx} \geq \pi^2.$$

Let $f(x)=\frac{\sin(\pi x)}{x}$ (extended continuously at 0) then $f(1)=0$. Therefore $g(x)=\sin(\pi x)$ and

$$\frac{\int_0^1 x^2(f'(x))^2\,dx}{\int_0^1 x^2(f(x))^2\,dx} = \frac{\int_0^1 (g'(x))^2\,dx}{\int_0^1 (g(x))^2\,dx} = \frac{\int_0^1 (\pi\cos(\pi x))^2\,dx}{\int_0^1 (\sin(\pi x))^2\,dx} = \pi^2.$$

Hence we may conclude that the desired minimum value is π^2. $\square$