Problem 12204
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Prove that the absolute value of the sum of the cosines of the four angles in a convex quadrilateral is less than $1/2$.

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Solution. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the four angles of the convex quadrilateral, we are going to show that

$$-\frac{1}{2} < \sum_{k=1}^{4} \cos(\alpha_k) < \frac{1}{2}.$$

We have that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2\pi$ and we may assume that $0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 < \pi$. It suffices to show that

$$\sum_{k=1}^{4} \cos(\alpha_k) < \frac{1}{2}. \tag{1}$$

Indeed let $\beta_i = \pi - \alpha_i$ for $i = 1, 2, 3, 4$, then $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 2\pi$, $0 < \beta_4 \leq \beta_3 \leq \beta_2 \leq \beta_1 < \pi$, and

$$-\sum_{k=1}^{4} \cos(\alpha_k) = \sum_{k=1}^{4} \cos(\beta_k) < \frac{1}{2} \implies -\frac{1}{2} < \sum_{k=1}^{4} \cos(\alpha_k)$$

In order to prove (1) we distinguish four cases.
1) If $\alpha_4 \leq \pi/2$ then $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \pi/2$ and $\sum_{k=1}^{4} \cos(\alpha_k) = 0 < 1/2$.
2) If $\alpha_3 \leq \pi/2 < \alpha_4$, since $x \to \cos(x)$ is concave in $[0, \pi/2]$, then by Jensen inequality,

$$\begin{aligned}\sum_{k=1}^{4} \cos(\alpha_k) &\leq 3\cos\left(\frac{\alpha_1 + \alpha_2 + \alpha_3}{3}\right) + \cos(\alpha_4) = 3\cos\left(\frac{2\pi - \alpha_4}{3}\right) + \cos(\alpha_4) \\ &< 3\cos\left(\frac{2\pi - \pi}{3}\right) + \cos(\pi) = \frac{3}{2} - 1 = \frac{1}{2}\end{aligned}$$

because $\alpha_4 \to 3\cos\left(\frac{2\pi - \alpha_4}{3}\right) + \cos(\alpha_4)$ is stricly increasing in $[\pi/2, \pi]$.
3) If $\alpha_2 \leq \pi/2 < \alpha_3$ then, again by Jensen inequality,

$$\begin{aligned}\sum_{k=1}^{4} \cos(\alpha_k) &\leq 2\cos\left(\frac{\alpha_1 + \alpha_2}{2}\right) + \cos(\alpha_3) + \cos(\alpha_4) = 2\cos\left(\frac{2\pi - (\alpha_3 + \alpha_4)}{2}\right) + \cos(\alpha_3) + \cos(\alpha_4) \\ &= -2\cos\left(\frac{\alpha_3 + \alpha_4}{2}\right) + \cos(\alpha_3) + \cos(\alpha_4) < -2\cos\left(\frac{\alpha_3 + \pi}{2}\right) + \cos(\alpha_3) - 1 \\ &< -2\cos\left(\frac{\pi/2 + \pi}{2}\right) + 0 - 1 = \sqrt{2} - 1 < \frac{1}{2}\end{aligned}$$

because, since $x \to \cos(x)$ is convex in $[\pi/2, \pi]$ it follows that $\alpha_4 \to -2\cos\left(\frac{\alpha_3 + \alpha_4}{2}\right) + \cos(\alpha_3) + \cos(\alpha_4)$ is strictly increasing in $[\pi/2, \pi]$, and $\alpha_3 \to -2\cos\left(\frac{\alpha_3 + \pi}{2}\right) + \cos(\alpha_3) - 1$ is strictly decreasing $[\pi/2, \pi]$.
4) If $\alpha_1 \leq \pi/2 < \alpha_2$ then

$$\begin{aligned}\sum_{k=1}^{4} \cos(\alpha_k) &= \cos(\alpha_2) + \cos(\alpha_3) + \cos(\alpha_4) + \cos(\alpha_2 + \alpha_3 + \alpha_4) \\ &= 2\cos\left(\frac{\alpha_2 + \alpha_3}{2}\right)\cos\left(\frac{\alpha_2 - \alpha_3}{2}\right) + 2\cos\left(\frac{2\alpha_4 + \alpha_2 + \alpha_3}{2}\right)\cos\left(\frac{\alpha_2 + \alpha_3}{2}\right) \\ &= 4\cos\left(\frac{\alpha_2 + \alpha_3}{2}\right)\cos\left(\frac{\alpha_3 + \alpha_4}{2}\right)\cos\left(\frac{\alpha_4 + \alpha_2}{2}\right) < 0 < \frac{1}{2}\end{aligned}$$

because $(\alpha_2 + \alpha_3)/2, (\alpha_3 + \alpha_4)/2, (\alpha_4 + \alpha_2)/2 \in (\pi/2, \pi)$. □

Remark. $1/2$ is the best constant: consider a convex quadrilateral with $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3 + t$ and $\alpha_4 = \pi - 3t$ then $\lim_{t \to 0^+} \sum_{k=1}^{4} \cos(\alpha_k) = 1/2$.