Problem 12203
(American Mathematical Monthly, Vol.127, October 2020)

Proposed by R. Tauraso (Italy).

Let m be a nonnegative integer, and let μ be the Möbius function on $\mathbb{Z}^+$, defined by setting $\mu(k)$ equal to $(-1)^r$ if k is the product of r distinct primes and equal to 0 if k has a square prime factor. Evaluate

$$\lim_{n\to\infty} \frac{1}{\ln^m(n)} \sum_{k=1}^{n} \frac{\mu(k)}{k} \ln^{m+1}\left(\frac{n}{k}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We will show by induction on m that the given limit is equal to $m+1$.
The case when $m=0$ is known (see H. Iwaniec *Lectures on the Riemann Zeta Function* p. 18):

$$\lim_{x\to\infty} \sum_{1\leq k\leq x} \frac{\mu(k)}{k} \log\left(\frac{x}{k}\right) = \lim_{x\to\infty} \log(x) \sum_{1\leq k\leq x} \frac{\mu(k)}{k} - \lim_{x\to\infty} \sum_{1\leq k\leq x} \frac{\mu(k)}{k} \log(k) = 0-(-1) = 1.$$

Note that if f is a function defined on $[1,+\infty)$ and $F(x) := \sum_{1\leq k\leq x} f\left(\frac{x}{k}\right)$ then the following Möbius inversion formula holds

$$\sum_{1\leq k\leq x} \mu(k) F\left(\frac{x}{k}\right) = \sum_{1\leq k\leq x} \mu(k) \sum_{1\leq j\leq x/k} f\left(\frac{x/k}{j}\right) = \sum_{1\leq n\leq x} f\left(\frac{x}{n}\right) \sum_{k|n} \mu(k) = f(x) \tag{1}$$

with $n=jk$. Now assume that $m\geq 1$ and let $f(x) = x\log^m(x)$ then

$$\begin{aligned}
\frac{F(x)}{x} &= \sum_{1\leq k\leq x} \frac{1}{k} \log^m\left(\frac{x}{k}\right) = x \sum_{1\leq k\leq x} \frac{1}{k} \sum_{j=0}^{m} \binom{m}{j} \log^{m-j}(x) \log^j(k) (-1)^j \\
&= \sum_{j=0}^{m} \binom{m}{j} (-1)^j \log^{m-j}(x) \sum_{1\leq k\leq x} \frac{\log^j(k)}{k} \\
&= \sum_{j=0}^{m} \binom{m}{j} (-1)^j \log^{m-j}(x) \left(\frac{\log^{j+1}(x)}{j+1} + c_j + O\left(\frac{\log^j(x)}{x}\right)\right) \\
&= \frac{1}{m+1} \sum_{j=0}^{m} \binom{m+1}{j+1} (-1)^j \log^{m+1}(x) + \sum_{j=0}^{m} \binom{m}{j} (-1)^j c_j \log^{m-j}(x) + O\left(\frac{\log^m(x)}{x}\right) \\
&= \frac{\log^{m+1}(x)}{m+1} + \sum_{j=0}^{m} \binom{m}{j} (-1)^j c_j \log^{m-j}(x) + O\left(\frac{\log^m(x)}{x}\right)
\end{aligned}$$

where we used the fact that for any $j\geq 0$ there exists $c_j\in\mathbb{R}$ such that for $x\to+\infty$,

$$\sum_{1\leq k\leq x} \frac{\log^j(k)}{k} = \frac{\log^{j+1}(x)}{j+1} + c_j + O\left(\frac{\log^j(x)}{x}\right).$$

Therefore, by (1),

$$\begin{aligned}
\log^m(x) &= \frac{f(x)}{x} = \frac{1}{x} \sum_{1\leq k\leq x} \mu(k) F\left(\frac{x}{k}\right) \\
&= \frac{1}{m+1} \sum_{1\leq k\leq x} \frac{\mu(k)}{k} \log^{m+1}\left(\frac{x}{k}\right) + \sum_{j=0}^{m} \binom{m}{j} (-1)^j c_j \sum_{1\leq k\leq x} \frac{\mu(k)}{k} \log^{m-j}\left(\frac{x}{k}\right) + O(1)
\end{aligned} \tag{2}$$

because for some constant C

$$\begin{aligned}\left|\sum_{1\leq k\leq x}\mu(k)\,O\left(\log^m(x/k)\right)\right| &\leq C\sum_{1\leq k\leq x}\log^m(x/k)\\ &\leq C\log^m(x)+C\int_1^x \log^m(x/t)\,dt\\ &\leq C\log^m(x)+Cx\int_1^{+\infty}\frac{\log^m(y)}{y^2}\,dy=O(x).\end{aligned}$$

Since $\sum_{1\leq k\leq x}\frac{\mu(k)}{k}=O(1)$ and, by the inductive step, for $0\leq j\leq m-1$,

$$\sum_{1\leq k\leq x}\frac{\mu(k)}{k}\log^{m-j}\left(\frac{x}{k}\right)=O(\log^{m-j-1}(x)),$$

after dividing by $\log^m(x)$ both sides of (2) we find

$$1=\frac{1}{m+1}\cdot\frac{1}{\log^m(x)}\sum_{1\leq k\leq x}\frac{\mu(k)}{k}\log^{m+1}\left(\frac{x}{k}\right)+\sum_{j=0}^{m}\binom{m}{j}(-1)^j c_j\cdot o(1)+o(1).$$

Therefore

$$\lim_{x\to\infty}\frac{1}{\log^m(x)}\sum_{1\leq k\leq x}\frac{\mu(k)}{k}\log^{m+1}\left(\frac{x}{k}\right)=m+1$$

and the proof is complete. □