Problem 12202
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Let V be a finite set of vectors in $\mathbb{R}^2$ such that $\sum_{v\in V}|v| = \pi$. Prove that there exists a subset U of V such that $\left|\sum_{v\in U} v\right| \geq 1$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. For any unit vector w, let

$$P_w(v) = \begin{cases} v \cdot w = |v|\cos(\theta) & \text{if } \cos(\theta) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

where θ is the angle between v and w. Then $|v| \geq P_w(v)$ and by letting $U_w = \{v \in V \;:\; P_w(v) > 0\}$, we find that

$$\left|\sum_{v\in U_w} v\right| \geq P_w\left(\sum_{v\in U_w} v\right) = \sum_{v\in U_w} P_w(v) = \sum_{v\in V} P_w(v).$$

Hence there exists a unit vector w such that $\left|\sum_{v\in U_w} v\right|$ is greater or equal the average value of $\sum_{v\in V} P_w(v)$ given by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{v\in V} P_w(v)\,d\theta = \frac{1}{2\pi}\sum_{v\in V}|v|\int_{-\pi/2}^{\pi/2}\cos(\theta)\,d\theta = \frac{1}{\pi}\sum_{v\in V}|v| = 1.$$

□

Remark. The lower bound 1 is the best possible one. Consider this set of evenly distributed two-dimensional vectors (with complex number notation)

$$V_n = \{v_k = \frac{\pi}{n}e^{i\frac{2\pi k}{n}} \;:\; k = 0,\ldots,n-1\}$$

where n is an even positive integer. Let $U_n = \{v_k \;:\; k = 0,\ldots,\frac{n}{2}-1\} \subset V_n$. Then $\sum_{v\in V_n}|v| = \pi$ and

$$\sum_{v\in U_n}|v| = \frac{\pi}{n}\left|\sum_{k=0}^{\frac{n}{2}-1} e^{i\frac{2\pi k}{n}}\right| = \frac{\pi}{n}\left|\frac{1-e^{i\pi}}{1-e^{i\frac{2\pi}{n}}}\right| = \frac{\pi}{n}\,\frac{2}{|e^{-i\frac{\pi}{n}} - e^{i\frac{\pi}{n}}|} = \frac{\pi/n}{\sin(\pi/n)} \to 1 \quad \text{as } n \to \infty.$$