Problem 12199
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Proposed by S. Sharma (India).

Prove

$$\int_0^\infty \frac{x\sinh(x)}{3+4\sinh^2(x)}\,dx = \frac{\pi^2}{24}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. By letting $e^{-x}=t$, we have that

$$\begin{aligned}
\int_0^\infty \frac{x\sinh(x)}{3+4\sinh^2(x)}\,dx &= -\frac{1}{2}\int_0^1 \frac{\log(t)(1/t-t)}{3+(1/t^2+t^2-2)}\,\frac{dt}{t} = -\frac{1}{2}\int_0^1 \frac{\log(t)(1-t^2)}{t^4+t^2+1}\,dt\\
&= -\frac{1}{2}\int_0^1 \frac{\log(t)(1-t^2)^2}{1-t^6}\,dt = -\frac{1}{2}\int_0^1 \log(t)(1-2t^2+t^4)\sum_{k=0}^\infty t^{6k}\,dt\\
&= \frac{1}{2}\left(-\int_0^1 \log(t)\sum_{k=0}^\infty t^{6k}\,dt + 2\int_0^1 \log(t)\sum_{k=0}^\infty t^{6k+2}\,dt - \int_0^1 \log(t)\sum_{k=0}^\infty t^{6k+4}\,dt\right)\\
&= \frac{1}{2}\left(-\sum_{k=0}^\infty \int_0^1 \log(t)t^{6k}\,dt + 2\sum_{k=0}^\infty \int_0^1 \log(t)t^{6k+2}\,dt - \sum_{k=0}^\infty \int_0^1 \log(t)t^{6k+4}\,dt\right)\\
&= \frac{1}{2}\left(\sum_{k=0}^\infty \frac{1}{(6k+1)^2} - 2\sum_{k=0}^\infty \frac{1}{(6k+3)^2} + \sum_{k=0}^\infty \frac{1}{(6k+5)^2}\right)\\
&= \frac{1}{2}\left(\sum_{k=0}^\infty \frac{1}{(2k+1)^2} - 3\sum_{k=0}^\infty \frac{1}{(6k+3)^2}\right)\\
&= \frac{1}{2}\left(1-\frac{3}{3^2}\right)\sum_{k=0}^\infty \frac{1}{(2k+1)^2} = \frac{1}{3}\sum_{k=0}^\infty \frac{1}{(2k+1)^2} = \frac{\pi^2}{24}
\end{aligned}$$

where the interchange of summation and integration is justified by Fubini's theorem and we applied

$$\int_0^1 \log(t)t^n\,dt = -\int_0^\infty xe^{-(n+1)x}\,dx = -\frac{1}{(n+1)^2}\int_0^\infty se^{-s}\,ds = -\frac{1}{(n+1)^2}.$$

□