Problem 12194
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Proposed by M. Tetiva (Romania).

Evaluate

$$\sum_{n=1}^{\infty}\left(H_n-\ln(n)-\gamma-\frac{1}{2n}\right)$$

where $H_n=\sum_{k=1}^{n}1/k$ *and* γ *is the Euler-Mascheroni constant.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that for $N\geq 1$,

$$\sum_{n=1}^{N}H_n=\sum_{k=1}^{N}\frac{1}{k}\sum_{n=k}^{N}1=\sum_{k=1}^{N}\frac{N+1-k}{k}=(N+1)H_N-N.$$

Hence, as $N\to\infty$,

$$\begin{aligned}\sum_{n=1}^{N}\left(H_n-\ln(n)-\gamma-\frac{1}{2n}\right)&=(N+1)H_N-N-\ln(N!)-\gamma N-\frac{H_N}{2}\\&=\left(N+\frac{1}{2}\right)\left(\ln(N)+\gamma+\frac{1}{2N}+o(1/N)\right)-(1+\gamma)N\\&\quad-\left(\ln(\sqrt{2\pi})+\frac{\ln(N)}{2}+N\ln(N)-N+o(1)\right)\\&=\frac{1+\gamma-\ln(2\pi)}{2}+o(1)\to\frac{1+\gamma-\ln(2\pi)}{2},\end{aligned}$$

where we used the following known asymptotic relations:

$$H_n=\ln(n)+\gamma+\frac{1}{2n}+o(1/n)\quad\text{and}\quad n!=\sqrt{2\pi n}\left(\frac{n}{e}\right)^n(1+o(1)).$$

□