Problem 12193
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Proposed by F. Stanescu (Romania).

Suppose that $f:[0,1]\to\mathbb{R}$ has a continuous third derivative and $f(0)=f(1)$. Prove

$$\left|\int_0^1 f'(x)x^{k-1}(1-x)^{k-1}\,dx\right|\leq\frac{(k-1)k!(k-1)!}{6(2k+1)!}\max_{0\leq x\leq 1}|f'''(x)|,$$

where k is a positive integer.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. By the integral form of the remainder in Taylor's Theorem, we have that

$$f'(x)=f'(0)+f''(0)x+\int_0^x (x-t)f'''(t)\,dt.$$

We multiply both sides by $b_k(x):=x^{k-1}(1-x)^{k-1}$ and we integrate over $[0,1]$ with respect to x:

$$\begin{aligned}\int_0^1 f'(x)x^{k-1}(1-x)^{k-1}\,dx&=B(k,k)f'(0)+B(k+1,k)f''(0)+\int_0^1 b_k(x)\int_0^x (x-t)f'''(t)\,dt\,dx\\&=B(k,k)\left(f'(0)+\frac{f''(0)}{2}\right)+\int_0^1\int_t^1 f'''(t)(x-t)b_k(x)\,dx\,dt\end{aligned}\tag{1}$$

where

$$B(n,m)=\int_0^1 t^{n-1}(1-t)^{m-1}\,dt=\frac{(n-1)!(m-1)!}{(n+m-1)!}$$

is the Beta function with n and m positive integers. For $k=1$ we have

$$0=f(1)-f(0)=\int_0^1 f'(x)\,dx=f'(0)+\frac{f''(0)}{2}+\int_0^x\int_t^1 f'''(t)(x-t)\,dx\,dt.$$

Therefore, after subtracting this equation multiplied by $B(k,k)$ from (1), we find

$$\int_0^1 f'(x)x^{k-1}(1-x)^{k-1}\,dx=-\int_0^1 f'''(t)g(t)dt$$

where $g(t)=\int_t^1(x-t)(B(k,k)-b_k(x))\,dx$. Note that $g'''(t)=-b_k'(t)\leq 0$ for $t\in[0,1/2]$ and therefore g' is concave in $[0,1/2]$. Since $g'(0)=g'(1/2)=0$, it follows that $g'(t)\geq 0$ for $t\in[0,1/2]$, which imply that $g(t)\geq g(0)=0$ holds first in $[0,1/2]$ and then in $[0,1]$ by the symmetry $g(t)=g(1-t)$.

Finally,

$$\left|\int_0^1 f'(x)x^{k-1}(1-x)^{k-1}\,dx\right|\leq\int_0^1 g(t)\,dt\cdot\max_{0\leq x\leq 1}|f'''(x)|=\frac{(k-1)k!(k-1)!}{6(2k+1)!}\max_{0\leq x\leq 1}|f'''(x)|$$

where

$$\begin{aligned}\int_0^1 g(t)\,dt&=\frac{B(k,k)}{6}-\int_0^1 b_k(x)\int_0^x(x-t)\,dt\,dx=\frac{B(k,k)}{6}-\frac{1}{2}\int_0^1 x^2 b_k(x)\\&=\frac{B(k,k)}{6}-\frac{B(k+2,k)}{2}=\frac{(k-1)k!(k-1)!}{6(2k+1)!}.\end{aligned}$$

□