**Problem 12192**
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Proposed by P. Kórus (Hungary).

*For any positive integers $a$, $b$ there is a real number $c > 0$ such that $(c, c^2)$ is a point on the graph of $y = x^2$ with minimum sum of distances to $(0, a)$ and $(0, b)$. Find all triples $(a, b, c)$ such that $c$ is a positive integer.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

*Solution.* We show that there are no such triples $(a, b, c)$.

Let $a$ and $b$ be positive integers. For any $x \in \mathbb{R}$, the sum of distances from $(x, x^2)$ to $(0, a)$ and $(0, b)$ is given by

$$f(x) := \sqrt{x^2 + (x^2 - a)^2} + \sqrt{x^2 + (x^2 - b)^2}.$$

The function $f$ is even, positive, differentiable, and $\lim_{|x|\to+\infty} f(x) = +\infty$. Hence there is $c \geq 0$ such that

$$f(c) = f(-c) = \min_{x\in\mathbb{R}} f(x).$$

Note that by considering the Taylor expansion of $f$ at $x = 0$,

$$\begin{aligned} f(x) &= a\sqrt{1 - \frac{2a-1}{a^2}x^2 + o(x^2)} + b\sqrt{1 - \frac{2b-1}{b^2}x^2 + o(x^2)} \\ &= a + b - \frac{1}{2}\left(\frac{2a-1}{a} + \frac{2b-1}{b}\right)x^2 + o(x^2), \end{aligned}$$

it follows that $f'(0) = 0$ and $f''(0) = -\left(\frac{2a-1}{a} + \frac{2b-1}{b}\right) < 0$ which imply that 0 is a local maximum point. Therefore the minimum point $c$ has to be positive. Moreover,

$$f'(x) = x \cdot \left(\frac{1 + 2(x^2 - a)}{\sqrt{x^2 + (x^2 - a)^2}} + \frac{1 + 2(x^2 - b)}{\sqrt{x^2 + (x^2 - b)^2}}\right),$$

and, since $f'(c) = 0$, after some manipulations, we find that the following equation holds

$$(4c^2 + 1)^2 = (4a - 1)(4b - 1).$$

It remains to prove that there is no positive integer $c$ which satisfies the above equation. We note that the positive integer factor $4a - 1$ has at least a prime divisor $p$ such that $p \equiv -1 \pmod 4$. Hence, if we assume that such positive integer $c$ exists, it follows that $p$ divides $4c^2 + 1$, that is $(2c)^2 \equiv -1 \pmod p$. After squaring we get $(2c)^4 \equiv 1 \pmod p$ and, by the Fermat's little theorem, we have that 4 is a divisor of $p - 1$, that is $p \equiv 1 \pmod 4$. Contradiction. □