Problem 12190
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Let ABC be a triangle, and let D, E, and F be points on AB, BC, and CA, respectively, such that AD, BE, and CF are concurrent at P. It is well known that if P is the orthocenter of ABC, then P is the incenter of DEF. Prove the converse.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Note that

$$\frac{AF}{FB} = \frac{\text{Area}(\triangle ADF)}{\text{Area}(\triangle FDB)} = \frac{AD \cdot FD \cdot \sin(\angle ADF)}{FD \cdot BD \cdot \sin(\angle FDB)} = \frac{AD \cdot \sin(\angle ADF)}{BD \cdot \sin(\angle FDB)},$$
$$\frac{CE}{EA} = \frac{\text{Area}(\triangle CDE)}{\text{Area}(\triangle EDA)} = \frac{CD \cdot ED \cdot \sin(\angle CDE)}{ED \cdot AD \cdot \sin(\angle EDA)} = \frac{CD \cdot \sin(\angle CDE)}{AD \cdot \sin(\angle EDA)}.$$

Since P is the incenter of DEF, we have that AD bisects the angle $\angle EDF$ and it follows that $\angle EDA = \angle ADF$. Hence, by Ceva's theorem,

$$1 = \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{AD \cdot \sin(\angle ADF)}{BD \cdot \sin(\angle FDB)} \cdot \frac{BD}{DC} \cdot \frac{CD \cdot \sin(\angle CDE))}{AD \cdot \sin(\angle EDA)} = \frac{\sin(\angle CDE)}{\sin(\angle FDB)},$$

which implies $\angle CDE = \angle FDB$ or $\angle CDE + \angle FDB = 180°$. Since

$$\angle CDE + \angle EDA + \angle ADF + \angle FDB = 180°,$$

it follows that $\angle CDE = \angle FDB$ and,

$$\angle CDE + \angle EDA = \angle ADF + \angle FDB = 90°,$$

that is, AD is perpendicular to BC. After repeating the same argument for BE and CF, we conclude that P is the orthocenter of the triangle ABC. □